

VINCIA Authors' Compendium

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A Final–Final Evolution Equations

A.1 Notation and Kinematic Relations

We denote the pre- and post-branching partons by $IK \rightarrow ijk$, respectively. For massless partons, note the relations (and notation):

$$E_{\text{cm}}^2 \equiv m_{\text{Ant}}^2 \equiv s_{IK} \equiv s_{ijk} = s_{ij} + s_{jk} + s_{ij} \quad (\text{A.1})$$

and

$$x_j = \frac{2E_j}{\sqrt{s_{IK}}} = 1 - \frac{s_{ik}}{s_{IK}}. \quad (\text{A.2})$$

For massive partons, we generally use the notation m for invariant masses and s for dot products, hence e.g., $m_{IK}^2 = (p_I + p_K)^2 = m_I^2 + m_K^2 + 2p_I \cdot p_K \equiv m_I^2 + m_K^2 + s_{IK}$, so that $s_{IK} \equiv 2p_I \cdot p_K$. The relation for massive particles is thus:

$$m_{\text{Ant}}^2 = m_{IK}^2 = m_{ijk}^2 = s_{IK} + m_I^2 + m_K^2 = s_{ij} + s_{jk} + s_{ik} + m_i^2 + m_j^2 + m_k^2. \quad (\text{A.3})$$

We define the scaled (dimensionless) invariants by

$$y_{ij} \equiv \frac{s_{ij}}{m_{IK}^2}. \quad (\text{A.4})$$

We also use the notation $\mu_i = m_i/m_{IK}$ for scaled masses.

A.2 Antenna Functions

For each antenna function, a full set of helicity-dependent antenna function contributions are implemented. For partons without helicity information, the unpolarised forms (summed over post-branching helicities and averaged over pre-branching ones) are used.

The functional forms given below omit colour and coupling factors. They are all normalised so that a factor $g_s^2 C_A = 4\pi\alpha_s C_A$ is appropriate in the leading-colour limit, with $C_A = 3$ replaced by $T_R = 1$ for gluon splittings.

Some of the helicity-dependent antenna functions would not be positive definite over the full branching phase space if only the singular terms were included. In particular the opposite-helicity gluon-emission antennnae therefore need to have

We here give only the forms for so-called “global” antenna functions, as sector antenna functions have not been fully implemented in the present VINCIA version.

The ij collinear limit of the functions can be studied by identifying $Q^2 = s_{ij} \rightarrow 0$ and

$$z_i = \frac{x_i}{x_i + x_j} = \frac{s_{IK} - s_{jk}}{s_{IK} + s_{ij}}, \quad (\text{A.5})$$

thus

$$s_{ij} \rightarrow Q^2 \quad (\text{A.6})$$

$$s_{jk} \rightarrow (1 - z_i) s_{IK} \quad (\text{A.7})$$

$$s_{ik} \rightarrow z_i s_{IK} \quad (\text{A.8})$$

and similarly for the ik collinear limit, with $i \leftrightarrow k$.

A.2.1 QQemitFF

The helicity average (for unpolarised partons) is:

$$a(q_I q_K \rightarrow q_i g_j q_k) = \frac{1}{m_{IK}^2} \left[\frac{2y_{ik}}{y_{ij}y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} - \frac{2\mu_K^2}{y_{jk}^2} + \frac{y_{jk}}{y_{ij}} + \frac{y_{ij}}{y_{jk}} + 1 \right] \quad (\text{A.9})$$

$$\begin{aligned} &= \frac{1}{m_{IK}^2} \left[\frac{(1-y_{ij})^2 + (1-y_{jk})^2}{y_{ij}y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} - \frac{2\mu_K^2}{y_{jk}^2} + 1 \right] \\ &= A_3^0 + 1. \end{aligned} \quad (\text{A.10})$$

Note: the A_3^0 function in GGG is derived from Z decay alone, while ours is the average of the Z^0 ones for $+/-/+$ ($J = 1$) parent configurations and the H^0 ones for $++/--$ ($J = 0$) ones. The difference between our antenna function and the GGG one is the $+1$ nonsingular term which is absent in the GGG A_3^0 .

The individual helicity contributions (for massless partons with specified helicities) are chosen such that all antenna functions remain positive definite over all of phase space.

$$a(++ \rightarrow +++) = \frac{1}{m_{IK}^2} \left[\frac{1}{y_{ij}y_{jk}} \right] \quad (\text{A.11})$$

$$a(++ \rightarrow +-+) = \frac{1}{m_{IK}^2} \left[\frac{(1-y_{ij})^2 + (1-y_{jk})^2 - 1}{y_{ij}y_{jk}} + 2 \right] \quad (\text{A.12})$$

$$a(+- \rightarrow +++) = \frac{1}{m_{IK}^2} \left[\frac{(1-y_{ij})^2}{y_{ij}y_{jk}} \right] \quad (\text{A.13})$$

$$a(+- \rightarrow +- -) = \frac{1}{m_{IK}^2} \left[\frac{(1-y_{jk})^2}{y_{ij}y_{jk}} \right]. \quad (\text{A.14})$$

Note that the sum of the two $++$ antenna functions has the same singularities as the sum of the two $+-$ ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

Note that, for a scalar decay, the helicity-flip one ($++ \rightarrow +-+$) has to go to zero on the hard boundary ($x_j = 1 - y_{ik} = y_{ij} + y_{jk} = 1$). In principle, one could (?) envision lifting this constraint (e.g., by adding a finite term that vanishes on the collinear boundaries, like $y_{ij}y_{jk}$) to account for t -channel processes which could produce a $++$ state with a higher total angular momentum, e.g., $J = 2$. Matrix-element corrections for scalar decay would then bring the matched result back to zero (which corresponds to the $+2$ finite term in the $+-+$ antenna), whiler MECs for t -channel processes could be non-zero. In the corresponding IF antenna, the finite term is instead chosen to be $+3 - y_{aj}^2 - y_{jk}^2$ which remains zero at the hard-collinear points but is nonzero along the diagonal.

A.2.2 QGemitFF

The helicity average (for unpolarised partons) is:

$$a(q_I g_K \rightarrow q_i g_j g_k) = \frac{1}{m_{IK}^2} \left[\frac{2y_{ik}}{y_{ij}y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} + \frac{y_{jk}}{y_{ij}} + \frac{y_{ij}(1-y_{ij})}{y_{jk}} + y_{ij} + \frac{y_{jk}}{2} \right] \quad (\text{A.15})$$

$$= \frac{1}{m_{IK}^2} \left[\frac{(1-y_{ij})^3 + (1-y_{jk})^2}{y_{ij}y_{jk}} - \frac{2\mu_I^2}{y_{ij}^2} + \frac{y_{ik} - y_{ij}}{y_{jk}} + 1 + y_{ij} + \frac{y_{jk}}{2} \right] \quad (\text{A.16})$$

$$\stackrel{\mu_I=0}{=} d_3^0 - \frac{5}{2} + 2y_{ij} + y_{jk} . \quad (\text{A.17})$$

Note 1: the D_3^0 function in GGG is derived from neutralino decay and contains a sum over both of the permutations of the gluons. Our function corresponds to the sub-antenna function d_3^0 from which it only differs by nonsingular terms.

Note 2: the singularity structure of the $qg \rightarrow qgg$ radiation function used in ARIADNE differs from ours by a term proportional to $(y_{ik} - y_{ij})$, which vanishes when summing over two neighbouring antennae (it is antisymmetric under interchange of the two gluons, j and k). Our parametrisation is chosen to agree with the GGG one which also has the property of minimising the sub-antenna contribution in the hard-collinear jk limit $y_{ij} \rightarrow 1$, corresponding to $x_k \rightarrow 1$. (Those configurations are then maximally populated by the soft limit of the neighbouring antenna, which minimises the problem of recoils producing disparate pT scales.)

The individual helicity contributions (for massless partons with specified helicities) are:

$$a(++ \rightarrow +++) = \frac{1}{m_{IK}^2} \left[\frac{1}{y_{ij}y_{jk}} + (1-\alpha)(1-y_{jk}) \left(\frac{1-2y_{ij}-y_{jk}}{y_{jk}} \right) \right] , \quad (\text{A.18})$$

$$a(++ \rightarrow +-+) = \frac{1}{m_{IK}^2} \left[\frac{(1-y_{ij})y_{ik}^2}{y_{ij}y_{jk}} \right] , \quad (\text{A.19})$$

$$a(+- \rightarrow +++) = \frac{1}{m_{IK}^2} \left[\frac{(1-y_{ij})^3}{y_{ij}y_{jk}} \right] , \quad (\text{A.20})$$

$$a(+- \rightarrow +-+) = \frac{1}{m_{IK}^2} \left[\frac{(1-y_{jk})^2}{y_{ij}y_{jk}} + (1-\alpha)(1-y_{jk}) \left(\frac{1-2y_{ij}-y_{jk}}{y_{jk}} \right) \right] . \quad (\text{A.21})$$

Note that the sum of the two ++ antenna functions has the same singularities as the sum of the two +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

A.2.3 GGemitFF

The helicity average (for unpolarised partons) is:

$$a(g_I g_K \rightarrow g_i g_j g_k) = \frac{1}{m_{IK}^2} \left[\frac{2y_{ik}}{y_{ij}y_{jk}} + \frac{y_{jk}(1-y_{jk})}{y_{ij}} + \frac{y_{ij}(1-y_{ij})}{y_{jk}} + \frac{1}{2}y_{ij} + \frac{1}{2}y_{jk} \right] \quad (\text{A.22})$$

$$= \frac{1}{m_{IK}^2} \left[\frac{(1-y_{ij})^3 + (1-y_{jk})^3}{y_{ij}y_{jk}} + \frac{y_{ik}-y_{ij}}{y_{jk}} + \frac{y_{ik}-y_{jk}}{y_{ij}} + 2 + \frac{1}{2}y_{ij} + \frac{1}{2}y_{jk} \right]. \quad (\text{A.23})$$

$$= f_3^0 - \frac{2}{3} - 2y_{ik} \quad (\text{A.24})$$

$$= f_3^{0'} - 1 - y_{ik} \quad (\text{A.25})$$

Note 1: the F_3^0 function in GGG is derived from Higgs decay and contains a sum over all three of the permutations of the gluons. Our function corresponds to the sub-antenna function f_3^0 from which it only differs by finite terms. The $f_3^{0'}$ function in the last line is an equivalent reparametrisation of f_3^0 which only differs by terms that cancel when summing over permutations. It is

$$f_3^{0'} = \frac{2y_{ik}}{y_{ij}y_{jk}} + \frac{y_{ik}y_{ij}}{y_{jk}} + \frac{y_{ik}y_{jk}}{y_{ij}} + 2 + y_{ij} + y_{jk}. \quad (\text{A.26})$$

Note 2: the sum of our ++ and − functions is equal to the GGG f_3^0 function modulo a reparametrisation which vanishes when summing over gluon permutations, hence the sum of those radiation functions is equal to F_3^0 .

Note 3: the singularity structure of the $gg \rightarrow ggg$ radiation function used in ARIADNE differs from ours by terms proportional to $(y_{ik} - y_{ij})/y_{jk}$ and $(y_{ik} - y_{jk})/y_{ij}$, which vanish when summing over neighbouring antennae (they are antisymmetric under interchange of gluons jk and ij respectively). Our parametrisation is chosen to agree with the GGG one which also has the property of minimising the sub-antenna contribution in the hard-collinear limits. (Those configurations are then maximally populated by the soft limit of the neighbouring antenna, which minimises the problem of recoils producing disparate pT scales.)

The individual helicity contributions are:

$$a(++ \rightarrow +++) = \frac{1}{m_{IK}^2} \left[\frac{1}{y_{ij}y_{jk}} + (1-\alpha) \left((1-y_{ij}) \frac{1-2y_{jk}-y_{ij}}{y_{ij}} + (1-y_{jk}) \frac{1-2y_{ij}-y_{jk}}{y_{jk}} \right) \right] \quad (\text{A.27})$$

$$a(++ \rightarrow +-+) = \frac{1}{m_{IK}^2} \left[\frac{y_{ik}^3}{y_{ij}y_{jk}} \right] \quad (\text{A.28})$$

$$a(+- \rightarrow +++-) = \frac{1}{m_{IK}^2} \left[\frac{(1-y_{ij})^3}{y_{ij}y_{jk}} + (1-\alpha)(1-y_{ij}) \frac{1-2y_{jk}}{y_{ij}} \right], \quad (\text{A.29})$$

$$a(+- \rightarrow +-- -) = \frac{1}{m_{IK}^2} \left[\frac{(1-y_{jk})^3}{y_{ij}y_{jk}} + (1-\alpha)(1-y_{jk}) \frac{1-2y_{ij}}{y_{jk}} \right]. \quad (\text{A.30})$$

Note that the sum of the two ++ antenna functions has the same singularities as the sum of the two +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

A.2.4 QGSplitFF and GGSplitFF

The QGSplit and GGSplit functions are chosen to be identical. (Note that for gg parents, the antenna function below is used for the splitting of the second gluon. The same expression is then applied, with swapped parents, to the splitting of the first gluon.)

For the generic case of a massive recoiler, $Xg \rightarrow X\bar{q}q$, the energy of the parent gluon in the Xg rest frame is $E_K = (m_{IK}^2 - m_I^2)/(2m_{IK}) = s_{IK}/2m_{IK}$. The energy fractions of the daughter quarks may then be defined as $x_j = E_j/E_K$ and similarly for x_k , which implies the following relations with the branching invariants,

$$x_j = 1 - \frac{1}{1 - \mu_I^2} y_{ik} \quad (\text{A.31})$$

$$x_k = 1 - \frac{1}{1 - \mu_I^2} y_{ij} , \quad (\text{A.32})$$

with $\mu_I = m_I/m_{IK}$. This remains valid for generic quark masses, $m_{j,k}$. Note also the kinematic limit for massive kinematics is $y_{ik} < 1 - \mu_I^2$ and similarly for y_{ij} , so that $x > 0$ over all of the physical phase space. In terms of these x fractions, the helicity average (for unpolarised partons) is:

$$a(X_I g_K \rightarrow X_i \bar{q}_j q_k) = \frac{1}{2m_{jk}^2} \left[(1 - x_j)^2 + (1 - x_k)^2 + \frac{2m_j^2}{m_{jk}^2} \right] \quad (\text{A.33})$$

In terms of the y_{ij} invariants, the expression is

$$a(X_I g_K \rightarrow X_i \bar{q}_j q_k) = \frac{1}{2m_{jk}^2} \left[\frac{y_{ik}^2 + y_{ij}^2}{(1 - \mu_I^2)^2} + \frac{2m_j^2}{m_{jk}^2} \right] \quad (\text{A.34})$$

The individual helicity contributions (for massless daughter quarks) are:

$$a(X+ \rightarrow X-+) = \frac{1}{2m_{jk}^2} \frac{y_{ik}^2}{(1 - \mu_I^2)^2} = \frac{(1 - x_j)^2}{2m_{jk}^2} \quad (\text{A.35})$$

$$a(X+ \rightarrow X+-) = \frac{1}{2m_{jk}^2} \frac{y_{ij}^2}{(1 - \mu_I^2)^2} = \frac{(1 - x_k)^2}{2m_{jk}^2} . \quad (\text{A.36})$$

Note that, in the first line, the quark (k) “inherits” the gluon helicity, while in the second line, the antiquark (j) inherits it. The $x \rightarrow 1$ limits are suppressed for the x that carries the opposite helicity to that of the splitting gluon.

A.3 Evolution Variables

The evolution variables considered in VINCIA for final-final antennae are the following [1, 2]:

$$Q_{\perp}^2 = N_{\perp} \frac{s_{ij}s_{jk}}{m_{IK}^2} = N_{\perp} y_{ij} y_{jk} m_{IK}^2 = N_{\perp} p_{\perp A}^2, \quad (\text{A.37})$$

$$m_D^2 = N_D \min(s_{ij}, s_{jk}) = N_D \min(y_{ij}, y_{jk}) m_{IK}^2, \quad (\text{A.38})$$

$$E^{*2} = \frac{(s_{ij} + s_{jk})^2}{m_{IK}^2} = (y_{ij} + y_{jk})^2 m_{IK}^2, \quad (\text{A.39})$$

$$m_{g^*}^2 = m_{jk}^2, \quad (\text{A.40})$$

with the arbitrary normalization factors $N_{\perp} \in [1, 4]$ and $N_D \in [1, 2]$, the invariant mass

$$m_{IK}^2 = (p_I + p_K)^2 = (p_i + p_j + p_k)^2, \quad (\text{A.41})$$

and the symbol s_{ij} defined as the dot product

$$s_{ij} \equiv 2p_i \cdot p_j = (p_i + p_j)^2 - m_i^2 - m_j^2 \stackrel{m=0}{=} m_{ij}^2. \quad (\text{A.42})$$

The maximum values that these evolution variables attain on the physical final-final antenna phase-space are:

$$Q_{\perp \max}^2 = \frac{N_{\perp}}{4} m_{IK}^2, \quad (\text{A.43})$$

$$m_{D \max}^2 = \frac{N_D}{2} m_{IK}^2, \quad (\text{A.44})$$

$$E_{\max}^{*2} = m_{IK}^2, \quad (\text{A.45})$$

$$m_{g^* \max}^2 = m_{IK}^2. \quad (\text{A.46})$$

Note on dimensionality: the dimensionless form of the evolution variable is $y = Q^2/m_{IK}^2$, with Q denoting the choice of evolution variable among the above possibilities. Throughout, we use the notation y for scaled dot products, $y_{ij} = s_{ij}/m_{IK}^2$, and y' for scaled invariant masses, $y'_{ij} = m_{ij}^2/m_{IK}^2$.

Note for m_D : the expressions below correspond to the branch with $y_{ij} < y_{jk}$ and hence will only generate branchings over half of phase space. For trial antenna functions symmetric in the invariants (specifically the soft-eikonal and hard-finite ones, see sec. A.6), the trial generation is done by multiplying the kernel by a factor 2 and randomly keeping or swapping the generated invariants. For the I - and K -collinear sector terms, we use that they are mutually related by $i \leftrightarrow k$, and hence an I -collinear term over all of phase space can be composed from an I -collinear one on the branch $y_{ij} < y_{jk}$ combined with a K -collinear one with swapped invariants on the complementary branch.

A.4 Zeta Definitions

The following choices of ζ are used:

$$\zeta_1 = \frac{y_{ij}}{y_{ij} + y_{jk}} \quad (\text{A.47})$$

$$\zeta_2 = y_{ij} \quad (\text{A.48})$$

$$\zeta_3 = y_{jk} . \quad (\text{A.49})$$

The final-final phase-space limits are, for Q_\perp :

$$\zeta_{1\pm}(Q_\perp^2) = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4}{N_\perp} \frac{Q_\perp^2}{m_{IK}^2}} \right) = \frac{1}{2} (1 \pm \sqrt{1 - 4 y_{ij} y_{jk}}) , \quad (\text{A.50})$$

$$\zeta_{2\pm}(Q_\perp^2) = \zeta_{1\pm}(Q_\perp^2) , \quad (\text{A.51})$$

$$\zeta_{3\pm}(Q_\perp^2) = \zeta_{1\pm}(Q_\perp^2) , \quad (\text{A.52})$$

for m_D :

$$\zeta_{1-}(m_D^2) = \frac{1}{2} , \quad (\text{A.53})$$

$$\zeta_{1+}(m_D^2) = 1 - \frac{m_D^2}{N_D m_{IK}^2} = 1 - y_{ij} , \quad (\text{A.54})$$

$$\zeta_{2-}(m_D^2) = \text{N/A} , \quad (\text{A.55})$$

$$\zeta_{2+}(m_D^2) = \text{N/A} , \quad (\text{A.56})$$

$$\zeta_{3-}(m_D^2) = \frac{m_D^2}{N_D m_{IK}^2} = y_{ij} , \quad (\text{A.57})$$

$$\zeta_{3+}(m_D^2) = 1 - \frac{m_D^2}{N_D m_{IK}^2} = 1 - y_{ij} , \quad (\text{A.58})$$

for E^* :

$$\zeta_\pm(E^{*2}) = \text{Special: see below} \quad (\text{A.59})$$

for m_{g^*} :

$$\zeta_{2-}(m_{q\bar{q}}^2) = 0 , \quad (\text{A.60})$$

$$\zeta_{2+}(m_{q\bar{q}}^2) = 1 - \frac{m_{g^*}^2}{m_{IK}^2} = 1 - y'_{jk} , \quad (\text{A.61})$$

using the definition $y'_{jk} = m_{jk}^2/m_{IK}^2$ in the last expression. Note that the phase-space limits for E^* coincide with the collinear limits. Integrations over any finite interval of E^* over the full allowed ζ range would therefore yield infinities. When using E^* -ordering, it is necessary to impose a hadronization cutoff in a complementary variable, such as Q_\perp or m_D . This cutoff then defines the ζ boundaries for the integrations.

A.5 Jacobians

The Jacobians for the transformation from the original LIPS variables, (s_{ij}, s_{jk}) , to the shower variables, (Q^2, ζ) , are written as a product of a normalization-and- Q -dependent piece and a ζ -dependent factor,

$$|J| = J_Q \times J_\zeta . \quad (\text{A.62})$$

They are, for Q_\perp^2 :

$$|J(Q_\perp^2, \zeta_1)| = \frac{1}{2N_\perp} \times \frac{m_{IK}^2}{\zeta_1(1 - \zeta_1)} , \quad (\text{A.63})$$

$$|J(Q_\perp^2, \zeta_2)| = \frac{1}{N_\perp} \times \frac{m_{IK}^2}{\zeta_2} , \quad (\text{A.64})$$

$$|J(Q_\perp^2, \zeta_3)| = \frac{1}{N_\perp} \times \frac{m_{IK}^2}{\zeta_3} , \quad (\text{A.65})$$

for m_D^2 :

$$|J(m_D^2, \zeta_1)| = \text{Not Used} , \quad (\text{A.66})$$

$$|J(m_D^2, \zeta_2)| = \text{N/A} , \quad (\text{A.67})$$

$$|J(m_D^2, \zeta_3)| = \frac{1}{N_D} \times m_{IK}^2 , \quad (\text{A.68})$$

for E^{*2} :

$$|J(E^{*2}, \zeta_1)| = \frac{1}{2} \times m_{IK}^2, \quad (\text{A.69})$$

$$|J(E^{*2}, \zeta_2)| = \frac{m_{IK}}{2\sqrt{E^{*2}}} \times m_{IK}^2, \quad (\text{A.70})$$

$$|J(E^{*2}, \zeta_3)| = \frac{m_{IK}}{2\sqrt{E^{*2}}} \times m_{IK}^2, \quad (\text{A.71})$$

for m_{g^*} :

$$|J(m_{q\bar{q}}^2, \zeta_2)| = 1 \times m_{IK}^2. \quad (\text{A.72})$$

A.6 Trial Functions

The following trial functions are available:

$$\text{Eikonal (soft)} : \hat{a}_E = \frac{1}{m_{IK}^2} \frac{2}{y_{ij} y_{jk}} \quad (\text{A.73})$$

$$\text{Constant (hard)} : \hat{a}_F = \frac{1}{m_{IK}^2} \quad (\text{A.74})$$

$$\text{I Collinear (sector)} : \hat{a}_I = \frac{1}{m_{IK}^2} \frac{2}{y_{ij}(1 - y_{jk})} \quad (\text{A.75})$$

$$\text{K Collinear (sector)} : \hat{a}_K = \frac{1}{m_{IK}^2} \frac{2}{y_{jk}(1 - y_{ij})} \quad (\text{A.76})$$

$$\text{K Splitting } (g \rightarrow q\bar{q}) : \hat{a}_S = \frac{1}{m_{q\bar{q}}^2} = \frac{1}{m_{IK}^2} \frac{1}{y'_{jk}}, \quad (\text{A.77})$$

where we emphasize that y'_{jk} in the last expression is defined by $y'_{jk} = m_{jk}^2/m_{IK}^2$.

A.7 Zeta Integrals

For a given trial antenna function, \hat{a} , the definition of the ζ integral is:

$$I_\zeta = \int_{\zeta_a}^{\zeta_b} d\zeta J_\zeta \hat{a} \quad (\text{A.78})$$

where $|J_\zeta|$ signifies the part of the Jacobian that only has ζ dependence (see above), and $\zeta_b > \zeta_a$ represents an arbitrary ζ interval. This interval will in general be larger than the physically

allowed one (trials generated outside the physical phase space will be rejected by a veto). We shall nevertheless still assume that all ζ values are at least inside the range $\zeta \in [0, 1]$.

The integration kernels are, for Q_\perp :

$$J_\zeta \hat{a}_{E,F}(Q_\perp^2, \zeta_1) = \frac{1}{(1 - \zeta_1)\zeta_1} , \quad (\text{A.79})$$

$$J_\zeta \hat{a}_I(Q_\perp^2, \zeta_3) = \frac{1}{(1 - \zeta_3)} , \quad (\text{A.80})$$

$$J_\zeta \hat{a}_K(Q_\perp^2, \zeta_2) = \frac{1}{(1 - \zeta_2)} , \quad (\text{A.81})$$

for m_D :

$$J_\zeta \hat{a}_E(m_D^2, \zeta_3) = \frac{1}{\zeta_3} , \quad (\text{A.82})$$

$$J_\zeta \hat{a}_F(m_D^2, \zeta_3) = 1 , \quad (\text{A.83})$$

$$J_\zeta \hat{a}_I(m_D^2, \zeta_3) = \frac{1}{(1 - \zeta_3)} , \quad (\text{A.84})$$

for E^* :

$$J_\zeta \hat{a}_E(E^{*2}, \zeta_1) = \frac{1}{\zeta_1^2} , \quad (\text{A.85})$$

$$J_\zeta \hat{a}_F(E^{*2}, \zeta_1) = \frac{1}{2} , \quad (\text{A.86})$$

$$(\text{A.87})$$

for m_{g^*} :

$$J_\zeta \hat{a}_S(m_{g^*}^2, \zeta_2) = 1 \quad (\text{A.88})$$

with integrals over the range $\zeta_a < \zeta_b$, for Q_\perp :

$$I_{\zeta E,F}(Q_\perp^2, \zeta_1) = \ln \left(\frac{\zeta_b(1 - \zeta_a)}{\zeta_a(1 - \zeta_b)} \right) , \quad (\text{A.89})$$

$$I_{\zeta I}(Q_\perp^2, \zeta_3) = \ln \left(\frac{1 - \zeta_a}{1 - \zeta_b} \right) , \quad (\text{A.90})$$

$$I_{\zeta K}(Q_\perp^2, \zeta_2) = \ln \left(\frac{1 - \zeta_a}{1 - \zeta_b} \right) , \quad (\text{A.91})$$

for m_D :

$$I_{\zeta E}(m_D^2, \zeta_3) = \ln \left(\frac{\zeta_b}{\zeta_a} \right) , \quad (\text{A.92})$$

$$I_{\zeta F}(m_D^2, \zeta_3) = \zeta_b - \zeta_a , \quad (\text{A.93})$$

$$I_{\zeta I}(m_D^2, \zeta_3) = \ln \left(\frac{1 - \zeta_a}{1 - \zeta_b} \right) , \quad (\text{A.94})$$

for E^* :

$$I_{\zeta E}(E^{*2}, \zeta_1) = \frac{1}{z_a} - \frac{1}{z_b} , \quad (\text{A.95})$$

$$I_{\zeta F}(E^{*2}, \zeta_1) = \zeta_b - \zeta_a , \quad (\text{A.96})$$

for m_{g^*} :

$$I_{\zeta S}(m_{g^*}^2, \zeta_2) = \zeta_b - \zeta_a . \quad (\text{A.97})$$

A.8 Evolution Integrals

The evolution integral, for a particular choice of Q and ζ , is defined as follows

$$\hat{\mathcal{A}}(Q_1^2, Q_2^2) = \int_{Q_2^2}^{Q_1^2} dQ^2 \frac{\mathcal{C} g_s^2}{16\pi^2 m_{IK}^2} J_Q(Q, \zeta) I_\zeta(Q, \zeta) , \quad (\text{A.98})$$

with \mathcal{C} the (trial) color factor (typically C_A for gluon emission and 1 for gluon splitting) and J_Q the non- ζ dependent part of the Jacobian, see eqs. (A.63) – (A.72).

Note: for massive partons, the phase-space factor should actually be larger: m_{IK} in the denominator should be replaced by the Källén function [3]:

$$m_{IK}^2 \rightarrow \lambda(m_{IK}^2, m_I^2, m_K^2) = m_{IK}^4 + m_I^4 + m_K^4 - 2(m_{IK}^2 m_I^2 + m_{IK}^2 m_K^2 + m_I^2 m_K^2) . \quad (\text{A.99})$$

For massive partons, this is taken care of during trial generation by applying an overall prefactor representing the phase-space volume and using the same integrals as shown here below.

We label the integrand in the above equation by

$$d\hat{\mathcal{A}} = \frac{\mathcal{C} g_s^2}{16\pi^2 s} J_Q(Q, \zeta) I_\zeta(Q, \zeta) , \quad (\text{A.100})$$

which takes the following specific forms, for Q_\perp :

$$d\hat{\mathcal{A}}_E(Q_\perp^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta E}(Q_\perp^2, \zeta_1) \frac{1}{Q_\perp^2}, \quad (\text{A.101})$$

$$d\hat{\mathcal{A}}_F(Q_\perp^2) = \frac{1}{2N_\perp} \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta F}(Q_\perp^2, \zeta_1) \frac{1}{m_{IK}^2}, \quad (\text{A.102})$$

$$d\hat{\mathcal{A}}_I(Q_\perp^2) = 2 \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta I}(Q_\perp^2, \zeta_3) \frac{1}{Q_\perp^2}, \quad (\text{A.103})$$

$$d\hat{\mathcal{A}}_K(Q_\perp^2) = 2 \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta K}(Q_\perp^2, \zeta_2) \frac{1}{Q_\perp^2}, \quad (\text{A.104})$$

for m_D :

$$d\hat{\mathcal{A}}_{E,I}(m_D^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} 2I_{\zeta E,I}(m_D^2, \zeta_3) \frac{1}{m_D^2}, \quad (\text{A.105})$$

$$d\hat{\mathcal{A}}_F(m_D^2) = \frac{1}{N_D} \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta F}(m_D^2, \zeta_3) \frac{1}{m_{IK}^2}, \quad (\text{A.106})$$

for E^* :

$$d\hat{\mathcal{A}}_E(E^{*2}) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta E}(E^{*2}, \zeta_1) \frac{1}{\sqrt{E^{*2}m_{IK}^2}}, \quad (\text{A.107})$$

$$d\hat{\mathcal{A}}_F(E^{*2}) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} \frac{1}{2} I_{\zeta F}(E^{*2}, \zeta_1) \frac{1}{m_{IK}^2}, \quad (\text{A.108})$$

for m_{g^*} :

$$d\hat{\mathcal{A}}_S(m_{g^*}^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta S}(m_{g^*}^2, \zeta_2) \frac{1}{m_{g^*}^2}. \quad (\text{A.109})$$

For a constant trial $\hat{\alpha}_s$, the evolution integrals are, for Q_\perp :

$$\hat{\mathcal{A}}_E^0(Q_{\perp 1}^2, Q_{\perp 2}^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta E} \ln \left(\frac{Q_{\perp 1}^2}{Q_{\perp 2}^2} \right), \quad (\text{A.110})$$

$$\hat{\mathcal{A}}_F^0(Q_{\perp 1}^2, Q_{\perp 2}^2) = \frac{1}{2N_\perp} \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta F} \frac{(Q_{\perp 1}^2 - Q_{\perp 2}^2)}{m_{IK}^2}, \quad (\text{A.111})$$

$$\hat{\mathcal{A}}_{I,K}^0(Q_{\perp 1}^2, Q_{\perp 2}^2) = 2 \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta I,K} \ln \left(\frac{Q_{\perp 1}^2}{Q_{\perp 2}^2} \right), \quad (\text{A.112})$$

for m_D :

$$\hat{\mathcal{A}}_{E,I}^0(m_{D1}^2, m_{D2}^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} 2I_{\zeta E,I} \ln \left(\frac{m_{D1}^2}{m_{D2}^2} \right), \quad (\text{A.113})$$

$$\hat{\mathcal{A}}_F^0(m_{D1}^2, m_{D2}^2) = \frac{1}{N_D} \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta F} \frac{(m_{D1}^2 - m_{D2}^2)}{m_{IK}^2}, \quad (\text{A.114})$$

for E^* :

$$\hat{\mathcal{A}}_E^0(E_1^{*2}, E_2^{*2}) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} 2I_{\zeta E} \frac{(\sqrt{E_1^{*2}} - \sqrt{E_2^{*2}})}{m_{IK}}, \quad (\text{A.115})$$

$$\hat{\mathcal{A}}_F^0(E_1^{*2}, E_2^{*2}) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} \frac{1}{2} I_{\zeta F} \frac{(E_1^{*2} - E_2^{*2})}{m_{IK}^2} \quad (\text{A.116})$$

for m_{g*} :

$$\hat{\mathcal{A}}_S^0(m_{g1}^2, m_{g2}^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta S} \ln \left(\frac{m_{g1}^2}{m_{g2}^2} \right). \quad (\text{A.117})$$

For a first-order running trial $\hat{\alpha}_s(Q^2)$,

$$\hat{\alpha}_s(Q_\perp^2) = \frac{1}{b_0 \ln \left(\frac{k_R^2 p_{\perp A}^2}{\Lambda^2} \right)} = \frac{1}{b_0 \ln \left(\frac{k_R^2 Q_\perp^2}{N_\perp \Lambda^2} \right)}, \quad (\text{A.118})$$

$$\hat{\alpha}_s(m_D^2) = \frac{1}{b_0 \ln \left(\frac{k_R^2 m_{\min}^2}{\Lambda^2} \right)} = \frac{1}{b_0 \ln \left(\frac{k_R^2 m_D^2}{N_D \Lambda^2} \right)}, \quad (\text{A.119})$$

$$\hat{\alpha}_s(m_{g*}^2) = \frac{1}{b_0 \ln \left(\frac{k_R^2 m_{g*}^2}{\Lambda^2} \right)}, \quad (\text{A.120})$$

with k_R an arbitrary scaling factor that includes the compound effect of any renormalization-scale prefactor choices and the optional translation between the MSbar and CMW schemes for Λ , the evolution integrals are, for Q_\perp :

$$\hat{\mathcal{A}}_E^1(Q_{\perp 1}^2, Q_{\perp 2}^2) = \frac{\mathcal{C} I_{\zeta E}}{4\pi b_0} \ln \left(\frac{\ln \left(\frac{k_R^2 Q_{\perp 1}^2}{N_\perp \Lambda^2} \right)}{\ln \left(\frac{k_R^2 Q_{\perp 2}^2}{N_\perp \Lambda^2} \right)} \right), \quad (\text{A.121})$$

$$\hat{\mathcal{A}}_F^1(Q_{\perp 1}^2, Q_{\perp 2}^2) = \text{Not Used (generates LogIntegrals)}, \quad (\text{A.122})$$

$$\hat{\mathcal{A}}_{I,K}^1(Q_{\perp 1}^2, Q_{\perp 2}^2) = 2 \frac{\mathcal{C} I_{\zeta I,K}}{4\pi b_0} \ln \left(\frac{\ln \left(\frac{k_R^2 Q_{\perp 1}^2}{N_\perp \Lambda^2} \right)}{\ln \left(\frac{k_R^2 Q_{\perp 2}^2}{N_\perp \Lambda^2} \right)} \right), \quad (\text{A.123})$$

for m_D :

$$\hat{\mathcal{A}}_{E,I}^1(m_{D1}^2, m_{D2}^2) = 2 \frac{CI_{\zeta E,I}}{4\pi b_0} \ln \left(\frac{\ln \left(\frac{k_R^2 m_{D1}^2}{N_D \Lambda^2} \right)}{\ln \left(\frac{k_R^2 m_{D2}^2}{N_D \Lambda^2} \right)} \right), \quad (\text{A.124})$$

$$\hat{\mathcal{A}}_F^1(m_{D1}^2, m_{D2}^2) = \text{Not Used (generates LogIntegrals)}, \quad (\text{A.125})$$

for m_g^* :

$$\hat{\mathcal{A}}_S^1(m_{g1}^2, m_{g2}^2) = \frac{CI_{\zeta S}}{4\pi b_0} \ln \left(\frac{\ln \left(\frac{k_R^2 m_{g1}^2}{\Lambda^2} \right)}{\ln \left(\frac{k_R^2 m_{g2}^2}{\Lambda^2} \right)} \right). \quad (\text{A.126})$$

A.9 Generation of Trial Evolution Scale

The trial Sudakov factor is defined as:

$$\hat{\Delta}(Q_1^2, Q_2^2) = \exp \left[-\hat{\mathcal{A}}(Q_1^2, Q_2^2) \right], \quad (\text{A.127})$$

and the next trial scale is found by solving the equation:

$$\mathcal{R} = \hat{\Delta}(Q^2, Q_{\text{new}}^2), \quad (\text{A.128})$$

for Q_{new} , with \mathcal{R} a random number distributed uniformly in the interval $\mathcal{R} \in [0, 1]$, and Q the current “restart scale”. For strongly ordered showers, the restart scale after an accepted trial branching is the evolution scale evaluated on the current parton configuration. For smoothly ordered showers, this restart scale is only used for antennae that are not color-adjacent to the branching that occurred; for the newly created antennae, and (optionally) for any color-adjacent ones, the restart scale is the respective antenna invariant masses¹.

For both strongly and smoothly ordered showers, the restart scale after a failed (vetoed) trial branching is the scale of the failed branching.

Note: to optimize event generation, trial scales can be saved and reused for any antennae whose flavors, spins, and invariant masses are preserved by the preceding branching step.

For constant trial $\hat{\alpha}_s$, the solutions for the next trial scale are, for Q_{\perp} :

$$Q_{\perp E \text{new}}^2 = Q_{\perp}^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta E}}}, \quad (\text{A.129})$$

$$Q_{\perp F \text{new}}^2 = Q_{\perp}^2 - m_{IK}^2 2N_{\perp} \frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta F}} \ln(1/\mathcal{R}), \quad (\text{A.130})$$

$$Q_{\perp I, K \text{new}}^2 = Q_{\perp}^2 \mathcal{R}^{\frac{1}{2} \frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta I, K}}}, \quad (\text{A.131})$$

¹This allows hard $2 \rightarrow n$ branchings to be generated inside the newly created antennae (and optionally within the color-adjacent ones) without disturbing the evolution of the rest of the event.

for m_D :

$$m_{DE,I\text{new}}^2 = m_D^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s} \frac{1}{2\mathcal{C}I_{\zeta E,I}}} , \quad (\text{A.132})$$

$$m_{DF\text{new}}^2 = m_D^2 - m_{IK}^2 N_D \frac{4\pi}{\hat{\alpha}_s} \frac{1}{\mathcal{C}I_{\zeta F}} \ln(1/\mathcal{R}) , \quad (\text{A.133})$$

for E^* :

$$E_{E\text{new}}^{*2} = \left(\sqrt{E^{*2}} - m_{IK} \frac{4\pi}{\hat{\alpha}_s} \frac{1}{\mathcal{C}I_{\zeta E}} \ln(1/\mathcal{R}) \right)^2 , \quad (\text{A.134})$$

$$E_{F\text{new}}^{*2} = E^{*2} - m_{IK}^2 \frac{4\pi}{\hat{\alpha}_s} \frac{1}{2\mathcal{C}I_{\zeta F}} \ln(1/\mathcal{R}) , \quad (\text{A.135})$$

for m_D :

$$m_{g^*S\text{new}}^2 = m_{g^*}^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s} \frac{1}{\mathcal{C}I_{\zeta S}}} . \quad (\text{A.136})$$

For a one-loop running trial $\hat{\alpha}_s(\mu_R^2)$, with $\mu_R^2 \propto Q^2$, the solutions for the next trial scale are, for Q_\perp :

$$\ln \left(\frac{k_R^2 Q_{\perp E\text{new}}^2}{N_\perp \Lambda^2} \right) = \mathcal{R}^{\frac{4\pi b_0}{\mathcal{C}I_{\zeta E}}} \ln \left(\frac{k_R^2 Q_\perp^2}{N_\perp \Lambda^2} \right) , \quad (\text{A.137})$$

$$\ln \left(\frac{k_R^2 Q_{\perp I,K\text{new}}^2}{N_\perp \Lambda^2} \right) = \mathcal{R}^{\frac{1}{2} \frac{4\pi b_0}{\mathcal{C}I_{\zeta E}}} \ln \left(\frac{k_R^2 Q_\perp^2}{N_\perp \Lambda^2} \right) , \quad (\text{A.138})$$

for m_D :

$$\ln \left(\frac{k_R^2 m_{DE,I\text{new}}^2}{N_D \Lambda^2} \right) = \mathcal{R}^{\frac{4\pi b_0}{2\mathcal{C}I_{\zeta E,I}}} \ln \left(\frac{k_R^2 m_D^2}{N_D \Lambda^2} \right) , \quad (\text{A.139})$$

for m_{g^*} :

$$\ln \left(\frac{k_R^2 m_{g^*S\text{new}}^2}{\Lambda^2} \right) = \mathcal{R}^{\frac{4\pi b_0}{\mathcal{C}I_{\zeta S}}} \ln \left(\frac{k_R^2 m_{g^*}^2}{\Lambda^2} \right) . \quad (\text{A.140})$$

A.10 Generation of Trial Zeta

The trial value for ζ is found by inverting the equation

$$\mathcal{R}_\zeta = \frac{I_\zeta(\zeta_{\min}, \zeta)}{I_\zeta(\zeta_{\min}, \zeta_{\max})} , \quad (\text{A.141})$$

where the boundary values $(\zeta_{\min}, \zeta_{\max})$ must be the same as those that were used to evaluate the I_ζ integrals during the generation of the trial scale above, i.e., they must correspond to the phase-space overestimate used for the trial generation. The forms of I_ζ are given for each evolution variable separately in eqs. (A.89)–(A.96).

For Q_\perp , the solutions to eq. (A.141) are:

$$\zeta_{1E,F}(\mathcal{R}) = \left[1 + \frac{1 - \zeta_{\min}}{\zeta_{\min}} \left(\frac{\zeta_{\min}(1 - \zeta_{\max})}{\zeta_{\max}(1 - \zeta_{\min})} \right)^{\mathcal{R}} \right]^{-1}, \quad (\text{A.142})$$

$$\zeta_{3I}(\mathcal{R}) = \zeta_{2K}(\mathcal{R}) = 1 - (1 - \zeta_{\min}) \left(\frac{1 - \zeta_{\max}}{1 - \zeta_{\min}} \right)^{\mathcal{R}}, \quad (\text{A.143})$$

for m_D :

$$\zeta_{3E}(\mathcal{R}) = \zeta_{\min} \left(\frac{\zeta_{\max}}{\zeta_{\min}} \right)^{\mathcal{R}}, \quad (\text{A.144})$$

$$\zeta_{3F}(\mathcal{R}) = \zeta_{\min} + \mathcal{R}(\zeta_{\max} - \zeta_{\min}), \quad (\text{A.145})$$

$$\zeta_{3I}(\mathcal{R}) = 1 - (1 - \zeta_{\min}) \left(\frac{1 - \zeta_{\max}}{1 - \zeta_{\min}} \right)^{\mathcal{R}}, \quad (\text{A.146})$$

for E^* :

$$\zeta_{1E}(\mathcal{R}) = \frac{\zeta_{\max}\zeta_{\min}}{\zeta_{\max} - \mathcal{R}(\zeta_{\max} - \zeta_{\min})}, \quad (\text{A.147})$$

$$\zeta_{1F}(\mathcal{R}) = \zeta_{\min} + \mathcal{R}(\zeta_{\max} - \zeta_{\min}), \quad (\text{A.148})$$

for m_{g^*} :

$$\zeta_{2S}(\mathcal{R}) = \zeta_{\min} + \mathcal{R}(\zeta_{\max} - \zeta_{\min}). \quad (\text{A.149})$$

A.11 Accept of Trial Zeta: Massless Phase-Space Boundaries

The generated value of ζ can now be compared to the limits imposed by the physical phase space at the generated value of Q and a rejection imposed if the generated ζ value falls outside the phase space, cf. eqs. (A.50)–(A.61).

A.12 Inverse Transforms

After a set of shower variables has been generated, the Q^2 and ζ choices must be inverted to reobtain the branching invariants, (s_{ij}, s_{jk}) , which are used to construct the kinematics of the

trial branching. These inversions are, for Q_\perp :

$$\begin{aligned}
& \mathbf{Q}_\perp^2 & \zeta_1 & \zeta_2 & \zeta_3 \\
y_{ij} &= \sqrt{\frac{\zeta_1}{1-\zeta_1}} \sqrt{\frac{Q_\perp^2}{N_\perp m_{IK}^2}} & \zeta_2 & \frac{1}{\zeta_3} \frac{Q_\perp^2}{N_\perp m_{IK}^2}, & (A.150) \\
y_{jk} &= \sqrt{\frac{1-\zeta_1}{\zeta_1}} \sqrt{\frac{Q_\perp^2}{N_\perp m_{IK}^2}} & \frac{1}{\zeta_2} \frac{Q_\perp^2}{N_\perp m_{IK}^2} & \zeta_3, &
\end{aligned}$$

for m_D (on $y_{ij} < y_{jk}$ branch):

$$\begin{aligned}
& \mathbf{m}_D^2 & \zeta_3 \\
y_{ij} &= \frac{m_D^2}{N_D}, & (A.151) \\
y_{jk} &= \zeta_3, &
\end{aligned}$$

for E^* :

$$\begin{aligned}
& \mathbf{E}^{*2} & \zeta_1 \\
y_{ij} &= \zeta_1 \sqrt{\frac{E^{*2}}{m_{IK}^2}}, & (A.152) \\
y_{jk} &= (1-\zeta_1) \sqrt{\frac{E^{*2}}{m_{IK}^2}}, &
\end{aligned}$$

for $m_{q\bar{q}}^2$:

$$\begin{aligned}
& \mathbf{m}_{q\bar{q}}^2 & \zeta_2 \\
y_{ij} &= \zeta_2, & (A.153) \\
y'_{jk} &= \frac{m_{q\bar{q}}^2}{m_{IK}^2}, &
\end{aligned}$$

A.13 Kinematics Construction

From a given set of invariants, (s_{ij}, s_{jk}) , the energies in the CM frame of the branching system can be constructed from

$$E_i = \frac{m^2 - m_{jk}^2 + m_i^2}{2m} \quad (A.154)$$

$$E_j = \frac{m^2 - m_{ik}^2 + m_j^2}{2m} \quad (A.155)$$

$$E_k = \frac{m^2 - m_{ij}^2 + m_k^2}{2m}, \quad (A.156)$$

with $m^2 = m_{IK}^2$ the invariant mass squared of the antenna. The relative angles between the momenta are given by:

$$\cos \theta_{ij} = \frac{2E_i E_j + m_i^2 + m_j^2 - m_{ij}^2}{2|p_i||p_j|} \quad (\text{A.157})$$

$$\cos \theta_{jk} = \frac{2E_j E_k + m_j^2 + m_k^2 - m_{jk}^2}{2|p_j||p_k|} . \quad (\text{A.158})$$

$$(\text{A.159})$$

The final orientation also depends on the choice of global recoil angle, ψ_{Ii} , which represents the angle between the pre- and post-branching partons I and i , around an axis perpendicular to the CM branching plane. Various specific forms can be chosen in the code, all of which must have the following collinear limits:

$$s_{jk} \rightarrow 0 : \psi_{Ii} \rightarrow 0 , \quad (\text{A.160})$$

$$s_{ij} \rightarrow 0 : \psi_{Kk} = \psi_{Ii} + \theta_{ik} - \pi \rightarrow 0 . \quad (\text{A.161})$$

A.14 Mass Corrections for Light Quarks (u,d,s,c,b)

By default, VINCIA treats all the lightest 5 quark flavours as massless. However, it is still possible to enable several corrections that approximate mass effects. The general treatment of massive quarks is documented in [3]. We here adapt this treatment to the case of massless kinematics, in the following way.

- Specify a mapping procedure that allows to translate an input parton state containing massive four-vectors for light-flavour quarks into an equivalent set of massless ones.
- Specify a reverse mapping that can be done at the end of the shower (or at an intermediate scale to change to a different number of massless flavours), to translate a set of massless partons into equivalent massive ones.
- Specify a procedure, consistent with the maps above, by which mass corrections can be applied in the context of the massless evolution.

A.14.1 Mapping from Massive to Massless Momenta

To map a set of partons containing massive four-vectors to equivalent massless ones, we map each 2-parton antenna in the input parton state to an equivalent massless one.

A.14.2 Mass Corrections

For a set of massless post-branching momenta, mass corrections are implemented in the following way, designed to fit with the mapping algorithm described above: first, identify the corresponding massive branching invariants by

$$s_{ij}^{\text{massless}} \rightarrow q_{ij}^2 = (2p_i \cdot p_j)^{\text{massive}} . \quad (\text{A.162})$$

The equivalent massive phase-space boundaries can then be checked by requiring positivity of the Gram determinant [3]:

$$4\Delta_3 = q_{ij}^2 q_{jk}^2 q_{ik}^2 - q_{ij}^4 m_k^2 - q_{jk}^4 m_i^2 - q_{ik}^4 m_j^2 + 4m_i^2 m_j^2 m_k^2 \geq 0, \quad (\text{A.163})$$

with $m_{i,j,k}$ the would-be physical masses of the post-branching partons.

A further level of refinement can be obtained by modifying the singularity structure of the massless antenna functions to take universal eikonal mass corrections into account (exact for soft gluon emissions) [3]:

$$a_{\text{massive}}^{\text{eikonal}}(Q_i, g_j, \bar{Q}_k) = a_{\text{massless}} - \frac{2m_i^2}{q_{ij}^4} - \frac{2m_k^2}{q_{jk}^4}, \quad (\text{A.164})$$

which can be applied as a multiplicative accept probability (with $P < 1$ since the corrections are negative) to all gluon-emission processes.

Finally, for branchings involving $g \rightarrow Q_j \bar{Q}_k$ splittings, we use the following multiplicative mass correction:

$$R_{Xg \rightarrow XQ\bar{Q}}(X_i, \bar{Q}_j, Q_k) = \frac{a_{\text{massless}} + \frac{m_Q^2}{m_{jk}^4}}{a_{\text{massless}}} \approx 1 + \frac{2m_Q^2}{m_{jk}^4} \frac{q_{IK}^4}{q_{ik}^4 + q_{ij}^4}. \quad (\text{A.165})$$

Since the sign of the mass correction is positive here (opposite to the case for gluon emission above), a headroom factor slightly greater than unity may be required to accommodate the enhanced splitting probability within the trial splitting overestimates.

A.14.3 Mapping from Massless to Massive Momenta

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B Initial–Initial Evolution Equations

B.1 Notation and Kinematic Relations

We denote the pre- and post-branching partons by $AB \rightarrow ajb$, respectively, for initial-state partons A and B evolving backwards to partons a , j , and b , with j in the final state and a and b in the initial state. (From the perspective of forwards evolution, partons a and b emit parton j .) For branchings involving initial-state partons, there is the additional aspect of the “hard system”, which we denote R (for “recoiler”) and which in general experiences a frame reinterpretation (Lorentz transformation = rotation + boost) as a consequence of the branching. Thus, the pre-branching system is $AB \rightarrow R$, with momentum conservation implying $p_A + p_B = p_R$, and the post-branching system is $ab \rightarrow j + r$ with $p_a + p_b - p_j = p_r$. For general masses, and using the notation $s_{12} \equiv 2p_1 \cdot p_2$, conservation of the invariant mass of the hard system ($m_R = m_r$) implies the relations

$$m_{ab}^2 \equiv s_{ab} + m_a^2 + m_b^2 = s_{AB} + s_{aj} + s_{jb} + m_A^2 + m_B^2 - m_j^2, \quad (\text{B.1})$$

$$s_{AB} + m_A^2 + m_B^2 = s_{ab} - s_{aj} - s_{jb} + m_a^2 + m_b^2 + m_j^2 . \quad (\text{B.2})$$

For dimensionless equivalents, we normalise by the largest invariant, s_{ab} , hence for example

$$1 + \mu_a^2 + \mu_b^2 = y_{AB} + y_{aj} + y_{jb} + \mu_A^2 + \mu_B^2 - \mu_j^2 . \quad (\text{B.3})$$

When a gluon is emitted into the final state we have $m_A = m_a$, $m_B = m_b$, and $m_j = 0$, hence

$$\text{Final-state gluon emission: } s_{ab} - s_{aj} - s_{jb} = s_{AB} . \quad (\text{B.4})$$

For quark creation (a.k.a. quark conversion: a quark backwards evolving to a gluon) on side a , we have $m_a = 0$, $m_A = m_q$, $m_j = m_q$, and $m_B = m_b$, hence also

$$g_a \rightarrow q_A \bar{q}_j: s_{ab} - s_{aj} - s_{jb} = s_{AB} , \quad (\text{B.5})$$

For gluon creation (aka gluon conversion; a gluon backwards evolving to a quark) on side a , we have $m_a = m_q$, $m_A = 0$, $m_j = m_q$, and $m_B = m_b$, hence

$$q_a \rightarrow g_A q_j: s_{ab} - s_{aj} - s_{jb} + 2m_q^2 = s_{AB} . \quad (\text{B.6})$$

Note that, although the relations above have been expressed for QCD branchings with general masses, the current initial-state shower implementation assumes all partons to be explicitly massless.

B.2 Crossing Relations

Compared to the FF case, crossing symmetry for the case when the recoiling partons i and k are crossed to be identified with a and b , respectively, implies

$$y_{ik} \rightarrow \frac{s_{ab}}{s_{AB}} \equiv 1/z \quad (\text{B.7})$$

$$y_{ij} \rightarrow -\frac{s_{aj}}{s_{AB}} \equiv -y_{aj}/z \quad (\text{B.8})$$

$$y_{jk} \rightarrow -\frac{s_{jb}}{s_{AB}} \equiv -y_{jb}/z . \quad (\text{B.9})$$

For crossings of partons $i \rightarrow a$ and $j \rightarrow b$,

$$y_{ik} \rightarrow -\frac{s_{ak}}{s_{AB}} \equiv -y_{ak}/z \quad (\text{B.10})$$

$$y_{ij} \rightarrow \frac{s_{ab}}{s_{AB}} \equiv 1/z \quad (\text{B.11})$$

$$y_{jk} \rightarrow -\frac{s_{kb}}{s_{AB}} \equiv -y_{kb}/z , \quad (\text{B.12})$$

where, if desired, one can obviously do a relabeling $k \rightarrow j$ for the II antennae such that the parton that remains in the final state is still labeled j .

For crossings of partons $j \rightarrow a$ and $k \rightarrow b$,

$$y_{ik} \rightarrow -\frac{s_{ib}}{s_{AB}} \equiv -y_{ib}/z \quad (\text{B.13})$$

$$y_{ij} \rightarrow -\frac{s_{ai}}{s_{AB}} \equiv -y_{ai}/z \quad (\text{B.14})$$

$$y_{jk} \rightarrow \frac{s_{ab}}{s_{AB}} \equiv 1/z. \quad (\text{B.15})$$

Note also that outgoing particles are mapped to incoming particles with the opposite helicities, so that, e.g., the II antenna function for $(\bar{q}_A^+ q_B^+ \rightarrow \bar{q}_a^+ g_j^+ q_b^+)$ is obtained from the FF one for $(q_I^- \bar{q}_K^- \rightarrow q_i^- g_j^+ \bar{q}_k^-)$.

Finally, we note that, in the current implementation of initial-state antenna functions, there are no II antenna functions for “emission into the initial state”; specifically, when both partons j and k are gluons, the crossing of ij is not used and similarly when both i and j are gluons. Taking the FF function for $qg \rightarrow qgg$ as an example, the function resulting from crossing quark i and gluon j is not used. (Instead, those terms are associated with emissions in the IF antenna between parton b and the recoiler, which produces the same colour structure). This is a choice of convention which effectively corresponds to a partial sectoring of the initial-state shower. It could be changed if future directions mandate it. Presently, the colour chain $a - b - j - R$ only contains the (IF) clustering of j into b and R but not one corresponding to “clustering” b into a and j (hence the name “emission into the initial state”). The abj antenna function and clustering could still be envisioned (with a corresponding subtraction of the terms present in the current bjR one); it would use an II kinematics map, possibly with fixed x_a . The choice of which mapping is associated with these terms affects whether they generate recoils (II) or not (IF). It could be an interesting student project to expand to “emissions into the initial state” and see how this affects recoil distributions, possibly in association with using an II map for the initial part of IF antennae.

B.3 Antenna Functions

B.3.1 QQemitII

The helicity average (for unpolarised partons) is:

$$a(\bar{q}_A q_B \rightarrow \bar{q}_a g_j q_b) = \frac{1}{s_{AB}} \left[\frac{2y_{AB}}{y_{aj}y_{jb}} + \frac{y_{jb}}{y_{aj}} + \frac{y_{aj}}{y_{jb}} + 1 \right] \quad (\text{B.16})$$

$$= \frac{1}{s_{AB}} \left[\frac{(1 - y_{aj})^2 + (1 - y_{jb})^2}{y_{aj}y_{jb}} + 1 \right]. \quad (\text{B.17})$$

The individual helicity contributions (for massless partons with specified helicities) are cho-

sen such that all antenna functions remain positive definite over all of phase space.

$$a(++ \rightarrow +++) = \frac{1}{s_{AB}} \frac{1}{y_{aj}y_{jb}}, \quad (\text{B.18})$$

$$a(++ \rightarrow +-+) = \frac{1}{s_{AB}} \frac{y_{AB}^2}{y_{aj}y_{jb}}, \quad (\text{B.19})$$

$$a(+ - \rightarrow +++ -) = \frac{1}{s_{AB}} \frac{(1 - y_{aj})^2}{y_{aj}y_{jb}}, \quad (\text{B.20})$$

$$a(+ - \rightarrow +- -) = \frac{1}{s_{AB}} \frac{(1 - y_{jb})^2}{y_{aj}y_{jb}}. \quad (\text{B.21})$$

Note that the sum of the two ++ antenna functions has the same singularities as the sum of the two +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

B.3.2 QGemitII

The helicity average (for unpolarised partons) is:

$$a(q_A g_B \rightarrow q_a g_j g_b) = \frac{1}{s_{AB}} \left[\frac{(1 - y_{aj})^3 + (1 - y_{jb})^2}{y_{aj}y_{jb}} + \frac{1 + y_{aj}^3}{y_{jb}(1 - y_{aj})} + 2 - y_{aj} - \frac{y_{jb}}{2} \right]. \quad (\text{B.22})$$

Note that the singular structure differs from the corresponding FF form by the term proportional to $1/(1 - y_{aj})$ which is a “sector” term (necessary since VINCIA does not include a sector for emission into the initial state). For comparison, the corresponding FF antenna is of the global type, and includes an antisymmetric $j \leftrightarrow k$ term which cancels when summing over neighbouring antennae.

The individual helicity contributions are crossings of FF sector antennae (see LLS):

$$a(++ \rightarrow ++++) = \frac{1}{s_{AB}} \left[\frac{1}{y_{aj}y_{jb}} \frac{1-y_{jb}}{1-y_{aj}-y_{jb}} \right] \quad (\text{B.23})$$

$$= \frac{1}{s_{AB}} \left[\frac{1}{y_{aj}y_{jb}} + \frac{1}{y_{jb}y_{AB}} \right] \quad (\text{B.24})$$

$$\xrightarrow{\text{sing}} \frac{1}{s_{AB}} \left[\frac{1}{y_{aj}y_{jb}} + \frac{1}{y_{jb}(1-y_{aj})} \right] . \quad (\text{B.25})$$

$$a(++ \rightarrow +-+) = \frac{1}{s_{AB}} \frac{1}{y_{aj}y_{jb}} \frac{y_{AB}^3}{1-y_{jb}} \quad (\text{B.26})$$

$$= \frac{y_{AB}^3}{s_{AB}} \left[\frac{1}{y_{aj}y_{jb}} + \frac{1}{y_{aj}(1-y_{jb})} \right] \quad (\text{B.27})$$

$$\xrightarrow{\text{sing}} \frac{1}{s_{AB}} \left[\frac{y_{AB}^3}{y_{aj}y_{jb}} + \frac{y_{AB}^2}{y_{aj}} \right] \quad (\text{B.28})$$

$$= \frac{1}{s_{AB}} \frac{(1-y_{aj})y_{AB}^2}{y_{aj}y_{jb}} , \quad (\text{B.29})$$

$$a(+ - \rightarrow +++ -) = \frac{1}{s_{AB}} \left[\frac{(1-y_{aj})^3}{y_{aj}y_{jb}} + \frac{1-y_{jb}-y_{aj}^2}{1-y_{jb}} \right] \quad (\text{B.30})$$

$$\xrightarrow{\text{sing}} \frac{1}{s_{AB}} \frac{(1-y_{aj})^3}{y_{aj}y_{jb}} , \quad (\text{B.31})$$

$$a(+ - \rightarrow +- -) = \frac{1}{s_{AB}} \frac{1}{y_{aj}y_{jb}} \frac{(1-y_{jb})^3}{1-y_{aj}-y_{jb}} \quad (\text{B.32})$$

$$= \frac{(1-y_{jb})^2}{s_{AB}} \left[\frac{1}{y_{aj}y_{jb}} + \frac{1}{y_{jb}} \frac{1}{1-y_{aj}-y_{jb}} \right] \quad (\text{B.33})$$

$$\xrightarrow{\text{sing}} \frac{1}{s_{AB}} \left[\frac{(1-y_{jb})^2}{y_{aj}y_{jb}} + \frac{1}{y_{jb}(1-y_{aj})} \right] . \quad (\text{B.34})$$

$$(\text{B.35})$$

Unlike the case for global antennae, there are also non-zero sector antenna functions involving helicity flips of the parent gluons; in a global language, these terms would be generated by the neighbouring antenna, but since VINCIA currently does not include a sector for “emission into the initial state”, the terms for which the helicity of incoming parton b is imparted to the outgoing gluon j , with B having positive helicity, must be included, leading to the additional sector terms that appear in the functions above (proportional to $1/y_{AB} = 1/z$), and to the two following

functions also being non-zero:

$$a(++ \rightarrow + - -) = \frac{1}{s_{AB}} \frac{y_{aj}^3}{y_{jb}(1 - y_{jb})} \frac{1}{1 - y_{aj} - y_{jb}} \quad (\text{B.36})$$

$$\xrightarrow{\text{sing}} \frac{1}{s_{AB}} \frac{y_{aj}^3}{y_{jb}(1 - y_{aj})} \quad (\text{B.37})$$

$$a(+ - \rightarrow + + +) = a(++ \rightarrow + - -) . \quad (\text{B.38})$$

Note: the $(1 - y_{jb})$ and $(1 - y_{aj})$ denominators are approximations of $1/y_{AB}$ denominators from the crossing. The corresponding $1/z$ singularity is not well suited for p_T resummation (the divergence sits at finite p_T even though it is of course not reached due to the requirement $x < 1$). The forms with $1/y_{AB}$ may still be useful for trial overestimates, since $y_{AB} = z$ over all of our phase space.

B.3.3 GGemitII

The helicity average (for unpolarised partons) is:

$$a(g_{AGB} \rightarrow g_a g_j g_b) = \frac{1}{s_{AB}} \left[\frac{(1 - y_{aj})^3 + (1 - y_{jb})^3}{y_{aj} y_{jb}} + \frac{1 + y_{aj}^3}{y_{jb}(1 - y_{aj})} + \frac{1 + y_{jb}^3}{y_{aj}(1 - y_{jb})} + 3 - \frac{3y_{aj}}{2} - \frac{3y_{jb}}{2} \right] . \quad (\text{B.39})$$

Note that this form is equivalent to the corresponding FF one, up to the terms proportional to $1/(1 - y_{aj})$ and $1/(1 - y_{jb})$ which are “sector” terms (necessary since VINCIA does not include a sector for emission into the initial state). For comparison, the corresponding FF antenna is of the global type, and includes an antisymmetric $j \leftrightarrow k$ term which cancels when summing over neighbouring antennae.

The individual helicity contributions are crossings of FF sector antennae (see LLS and the comments and limits taken in the QGemitII section; here we just give the final forms):

$$a(++ \rightarrow + + +) = \frac{1}{s_{AB}} \left[\frac{1}{y_{aj} y_{jb}} + \frac{1}{y_{jb}(1 - y_{aj})} + \frac{1}{y_{aj}(1 - y_{jb})} \right] , \quad (\text{B.40})$$

$$a(++ \rightarrow + - +) = \frac{1}{s_{AB}} \frac{y_{AB}^3}{y_{aj} y_{jb}} , \quad (\text{B.41})$$

$$a(+ - \rightarrow + + -) = \frac{1}{s_{AB}} \left[\frac{(1 - y_{aj})^3}{y_{aj} y_{jb}} + \frac{1}{y_{aj}(1 - y_{jb})} \right] , \quad (\text{B.42})$$

$$a(+ - \rightarrow + - -) = \frac{1}{s_{AB}} \left[\frac{(1 - y_{jb})^3}{y_{aj} y_{jb}} + \frac{1}{y_{jb}(1 - y_{aj})} \right] \quad (\text{B.43})$$

$$(\text{B.44})$$

The nonzero functions involving helicity flips of the parent gluons (see the comments in the

QGEmitII section) are:

$$a(++ \rightarrow +--) = \frac{1}{s_{AB}} \frac{y_{aj}^3}{y_{jb}(1-y_{aj})}, \quad (\text{B.45})$$

$$a(++ \rightarrow --+) = \frac{1}{s_{AB}} \frac{y_{jb}^3}{y_{aj}(1-y_{jb})}, \quad (\text{B.46})$$

$$a(+- \rightarrow +++) = a(++ \rightarrow +--) , \quad (\text{B.47})$$

$$a(+- \rightarrow ---) = a(++ \rightarrow --+) . \quad (\text{B.48})$$

Note: the $(1 - y_{jb})$ and $(1 - y_{aj})$ denominators are approximations of $1/y_{AB}$ denominators from the crossing. The corresponding $1/z$ singularity is not well suited for p_T resummation (the divergence sits at finite p_T even though it is of course not reached due to the requirement $x < 1$). The difference are the terms that are resummed by programs like HEJ.

B.3.4 QXsplitII

Quark (or antiquark) in the initial state backwards-evolving into a gluon and emitting an anti-quark (quark) into the final state. Can be obtained by crossing the QQemitFF functions, with final-state parton j as one of the crossed partons and relabeling. (Note hence this function does provide an example of a crossing that corresponds to “emission into the initial state”).

The helicity average (for unpolarised partons) is

$$a(q_A X_B \rightarrow g_a \bar{q}_j X_b) = \frac{1}{s_{AB}} \frac{y_{AB}^2 + (1 - y_{AB})^2}{y_{aj}} \quad (\text{B.49})$$

The individual helicity contributions are:

$$a(+X \rightarrow +-X) = \frac{1}{s_{AB}} \frac{y_{AB}^2}{y_{aj}} \quad (\text{B.50})$$

$$a(+X \rightarrow --X) = \frac{1}{s_{AB}} \frac{(1 - y_{AB})^2}{y_{aj}} \quad (\text{B.51})$$

$$a(-X \rightarrow -+X) = \frac{1}{s_{AB}} \frac{y_{AB}^2}{y_{aj}} \quad (\text{B.52})$$

$$a(-X \rightarrow ++X) = \frac{1}{s_{AB}} \frac{(1 - y_{AB})^2}{y_{aj}} \quad (\text{B.53})$$

$$(\text{B.54})$$

Note that the numerators express helicity conservation. The same expressions hold for backwards evolution of an antiquark, i.e. for $\bar{q}_A X_B$.

Note, the crossing of antenna function for massive quarks is:

$$a(Q_A X_B \rightarrow g_a \bar{Q}_j X_b) = \frac{1}{s_{AB}} \frac{y_{AB}^2 + (1 - y_{AB})^2}{y_{aj}} + \frac{2m_j^2}{s_{aj}^2} \quad (\text{B.55})$$

$$= \frac{1}{s_{AB}} \left[\frac{y_{AB}^2 + (1 - y_{AB})^2}{y_{aj}} + \frac{2\mu_j^2}{y_{AB}} \frac{y_{AB}^2}{y_{aj}^2} \right] \quad (\text{B.56})$$

B.3.5 GXconvII

Gluon in the initial state backwards-evolving into a quark (or antiquark) and emitting a quark (antiquark) into the final state. These functions can be obtained by crossing either parton j or k in the QGsplitFF functions.

The helicity average (for unpolarised partons) is

$$a(g_A X_B \rightarrow q_a q_j X_b) = \frac{1}{s_{AB}} \frac{1 + (1 - y_{AB})^2}{2y_{aj}(1 - y_{jb})}. \quad (\text{B.57})$$

The individual helicity contributions are:

$$a(+X \rightarrow ++X) = \frac{1}{s_{AB}} \frac{1}{2y_{aj}(1 - y_{jb})} \quad (\text{B.58})$$

$$a(+X \rightarrow --X) = \frac{1}{s_{AB}} \frac{(1 - y_{AB})^2}{2y_{aj}(1 - y_{jb})} \quad (\text{B.59})$$

$$a(-X \rightarrow --X) = \frac{1}{s_{AB}} \frac{1}{2y_{aj}(1 - y_{jb})} \quad (\text{B.60})$$

$$a(-X \rightarrow ++X) = \frac{1}{s_{AB}} \frac{(1 - y_{AB})^2}{2y_{aj}(1 - y_{jb})} \quad (\text{B.61})$$

$$(\text{B.62})$$

Note that the numerators express helicity conservation. The same expressions hold for backwards to an antiquark, i.e. for $g_A X_B \rightarrow \bar{q}_a \bar{q}_j X_b$.

Note 2: the $(1 - y_{jb})$ denominators are approximations of $1/y_{AB}$ denominators from the crossing. The corresponding $1/z$ singularity is not well suited for p_T resummation (the divergence sits at finite p_T even though it is of course not reached due to the requirement $x < 1$).

Note 3: for excitation of a massive quark, the crossing of the antenna function is

$$a(g_A X_B \rightarrow q_a q_j X_b) = \frac{1}{2s_{AB}(y_{aj} - 2\mu^2)} \left[\frac{1 + (1 - y_{AB})^2}{y_{AB}} - \frac{2\mu^2 y_{AB}}{y_{aj} - 2\mu^2} \right]. \quad (\text{B.63})$$

B.4 Evolution Variables

The evolution variables we use are

$$Q_\perp^2 = \frac{s_{aj}s_{jb}}{s_{ab}} = \frac{s_{aj}s_{jb}}{s_{AB} + s_{aj} + s_{jb}}, \quad (\text{B.64})$$

$$Q_A^2 = s_{aj}, \quad (\text{B.65})$$

$$Q_B^2 = s_{jb}. \quad (\text{B.66})$$

Phase Space Boundaries: With $s_{ab} = s_{AB} + s_{aj} + s_{jb}$ and $s_{ab} \leq s$, the phase space boundaries are $0 \leq s_{aj} + s_{jb} \leq s - s_{AB}$ and for the single branching invariants $0 \leq s_{aj} \leq s - s_{AB}$ and

$0 \leq s_{jb} \leq s - s_{AB}$. The maxima of the evolution variables are

$$Q_{\perp \max}^2 = \frac{1}{4} \frac{(s - s_{AB})^2}{s}, \quad (\text{B.67})$$

$$Q_{A \max}^2 = s - s_{AB}, \quad (\text{B.68})$$

$$Q_{B \max}^2 = s - s_{AB}. \quad (\text{B.69})$$

B.5 Zeta Definitions

The choices are

$$\zeta_1 = \frac{s_{aj}}{s_{ab}} = \frac{s_{aj}}{s_{AB} + s_{aj} + s_{jb}} = y_{aj}, \quad (\text{B.70})$$

$$\zeta_2 = \frac{s_{aj}}{s_{AB}} = \frac{s_{ab}}{s_{AB}} y_{aj}, \quad (\text{B.71})$$

$$\zeta_3 = \frac{s_{jb}}{s_{AB}}, \quad (\text{B.72})$$

$$\zeta_4 = \frac{s_{ab}}{s_{AB}}. \quad (\text{B.73})$$

The integration boundaries for the ζ variables are

$$\zeta_{1\pm}(Q_{\perp}^2) = \frac{1}{2s} \left(s - s_{AB} \pm \sqrt{(s - s_{AB})^2 - 4Q_{\perp}^2 s} \right), \quad (\text{B.74})$$

$$\zeta_{2\pm}(Q_{\perp}^2) = \frac{1}{2s_{AB}} \left(s - s_{AB} \pm \sqrt{(s - s_{AB})^2 - 4Q_{\perp}^2 s} \right), \quad (\text{B.75})$$

$$\zeta_{3\pm}(Q_{\perp}^2) = \frac{1}{2s_{AB}} \left(s - s_{AB} \pm \sqrt{(s - s_{AB})^2 - 4Q_{\perp}^2 s} \right), \quad (\text{B.76})$$

$$\zeta_{4-}(Q_{A/B}^2) = \frac{s_{AB} + Q_{A/B}^2}{s_{AB}} \quad \zeta_{4+}(Q_{A/B}^2) = \frac{s}{s_{AB}}. \quad (\text{B.77})$$

B.6 Jacobians

The Jacobians for the transformation from the phase-space variables, (s_{aj}, s_{jb}) , to the shower variables, (Q_E, ζ) , are

$$|J(Q_{\perp}^2, \zeta_1)| = s_{ab} \frac{1}{\zeta_1(1 - \zeta_1)}, \quad (\text{B.78})$$

$$|J(Q_{\perp}^2, \zeta_2)| = s_{ab}^2 \frac{1}{s_{aj}} \frac{1}{1 + \zeta_2}, \quad (\text{B.79})$$

$$|J(Q_{\perp}^2, \zeta_3)| = s_{ab}^2 \frac{1}{s_{jb}} \frac{1}{1 + \zeta_3}, \quad (\text{B.80})$$

$$|J(Q_A^2, \zeta_4)| = s_{AB}, \quad (\text{B.81})$$

$$|J(Q_B^2, \zeta_4)| = s_{AB}. \quad (\text{B.82})$$

B.7 Trial Functions

The following II trial functions are used:

$$\hat{a}_{\text{soft}} = \frac{1}{s_{AB}} \frac{2s_{ab}^2}{s_{aj}s_{jb}} = \frac{1}{s_{AB}} \frac{2}{y_{aj}y_{jb}}, \quad (\text{B.83})$$

$$\hat{a}_{\text{coll A}} = 2 \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{aj}} = \frac{1}{s_{AB}} \frac{2}{y_{aj}(1 - y_{aj} - y_{jb})}, \quad (\text{B.84})$$

$$\hat{a}_{\text{coll B}} = 2 \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{jb}} = \frac{1}{s_{AB}} \frac{2}{y_{jb}(1 - y_{aj} - y_{jb})}, \quad (\text{B.85})$$

$$a_{\text{split A}} = \frac{1}{s_{AB}} \left(-2 \frac{s_{jb}s_{AB}}{s_{aj}(s_{ab} - s_{aj})} + \frac{s_{ab}}{s_{aj}} \right) \Rightarrow \hat{a}_{\text{split A}} = \frac{s_{ab}}{s_{AB}} \frac{1}{s_{aj}}, \quad (\text{B.86})$$

$$a_{\text{split B}} = \frac{1}{s_{AB}} \left(-2 \frac{s_{aj}s_{AB}}{s_{jb}(s_{ab} - s_{jb})} + \frac{s_{ab}}{s_{jb}} \right) \Rightarrow \hat{a}_{\text{split B}} = \frac{s_{ab}}{s_{AB}} \frac{1}{s_{jb}}, \quad (\text{B.87})$$

$$a_{\text{conv A}} = \frac{1}{2s_{aj}} \frac{s_{jb}^2 + s_{ab}^2}{s_{AB}^2} \Rightarrow \hat{a}_{\text{conv A}} = \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{aj}}, \quad (\text{B.88})$$

$$a_{\text{conv B}} = \frac{1}{2s_{jb}} \frac{s_{aj}^2 + s_{ab}^2}{s_{AB}^2} \Rightarrow \hat{a}_{\text{conv B}} = \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{jb}}. \quad (\text{B.89})$$

Note that the overestimate of the soft eikonal term already includes the collinear singularities of the quarks.

B.8 Integration Kernels

The overestimate of the evolution integral is

$$\hat{A}(Q_{EF}^2, Q_{E\text{new}}^2) = \int_{Q_{E\text{new}}^2}^{Q_{EF}^2} \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \hat{a} \hat{R}_f |J| dQ_E^2 d\zeta \frac{d\phi}{2\pi}. \quad (\text{B.90})$$

The integration kernels for the Q_E^2 integration are

1. Soft with Q_{\perp}^2 and ζ_1 :

$$\begin{aligned} d\hat{\mathcal{A}}_{\text{soft}}(Q_{\perp}^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{1}{s_{AB}} \frac{2s_{ab}^2}{s_{aj}s_{jb}} s_{ab} \frac{1}{\zeta_1(1 - \zeta_1)} \hat{R}_f dQ_{\perp}^2 d\zeta_1 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{dQ_{\perp}^2}{Q_{\perp}^2} \frac{d\zeta_1}{\zeta_1(1 - \zeta_1)} \end{aligned} \quad (\text{B.91})$$

2. A gluon collinear with Q_{\perp}^2 and ζ_3 (similar B gluon collinear with ζ_2):

$$\begin{aligned} d\hat{\mathcal{A}}_{\text{coll A}}(Q_{\perp}^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} 2 \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{aj}} s_{ab}^2 \frac{1}{s_{jb}} \frac{1}{1 + \zeta_3} \hat{R}_f dQ_{\perp}^2 d\zeta_3 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_{xf} \frac{dQ_{\perp}^2}{Q_{\perp}^2} \frac{d\zeta_3}{1 + \zeta_3} \end{aligned} \quad (\text{B.92})$$

3. A gluon splitting

(a) with Q_\perp^2 and ζ_3 (similar B gluon splitting with ζ_2):

$$\begin{aligned} d\hat{\mathcal{A}}_{\text{split A}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{s_{ab}}{s_{AB}} \frac{1}{s_{aj}} s_{ab}^2 \frac{1}{s_{jb}} \frac{1}{1+\zeta_3} \hat{R}_f dQ_\perp^2 d\zeta_3 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{dQ_\perp^2}{Q_\perp^2} \frac{d\zeta_3}{1+\zeta_3} \end{aligned} \quad (\text{B.93})$$

(b) with Q_A^2 and ζ_4 (similar B gluon splitting with Q_B^2 and ζ_4):

$$\begin{aligned} d\hat{\mathcal{A}}_{\text{split A}}(Q_A^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{s_{ab}}{s_{AB}} \frac{1}{s_{aj}} s_{AB} \hat{R}_f dQ_A^2 d\zeta_4 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{s_{AB}}{s_{ab}} \frac{dQ_A^2}{Q_A^2} d\zeta_4 \left(\frac{x_a x_b}{x_A x_B} \frac{1}{\zeta_4} \right)^{1+\alpha} \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{x^\alpha f} \frac{dQ_A^2}{Q_A^2} \frac{d\zeta_4}{\zeta_4^{1+\alpha}} \end{aligned} \quad (\text{B.94})$$

4. A conversion

(a) with Q_\perp^2 and ζ_3 (similar B conversion with ζ_2):

$$\begin{aligned} d\hat{\mathcal{A}}_{\text{conv A}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{aj}} s_{ab}^2 \frac{1}{s_{jb}} \frac{1}{1+\zeta_3} \hat{R}_f dQ_\perp^2 d\zeta_3 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{xf} \frac{dQ_\perp^2}{Q_\perp^2} \frac{d\zeta_3}{1+\zeta_3} \end{aligned} \quad (\text{B.95})$$

(b) with Q_A^2 and ζ_4 (similar B conversion with Q_B^2 and ζ_4):

$$\begin{aligned} d\hat{\mathcal{A}}_{\text{conv A}}(Q_A^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AB}}{s_{ab}^2} \frac{s_{ab}^2}{s_{AB}^2} \frac{1}{s_{aj}} s_{AB} \hat{R}_f dQ_A^2 d\zeta_4 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{dQ_A^2}{Q_A^2} d\zeta_4 \left(\frac{x_a x_b}{x_A x_B} \frac{1}{\zeta_4} \right)^\alpha \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{x^\alpha f} \frac{dQ_A^2}{Q_A^2} \frac{d\zeta_4}{\zeta_4^\alpha} \end{aligned} \quad (\text{B.96})$$

Note that

$$\hat{R}_{xf} = \frac{x_a x_b}{x_A x_B} \hat{R}_f = \frac{s_{ab}}{s_{AB}} \hat{R}_f \quad (\text{B.97})$$

and that we introduced a general factors α to control the PDF ratio better.

B.9 Zeta Integrals and Generation of Trial Zeta

We have five different ζ integrals to solve,

$$I_{\zeta_1} = \int_{\zeta_a}^{\zeta_b} d\zeta_1 \frac{1}{\zeta_1(1-\zeta_1)} = \ln \left(\frac{\zeta_1}{1-\zeta_1} \right) \Big|_{\zeta_a}^{\zeta_b} = \ln \left(\frac{\zeta_b(1-\zeta_a)}{\zeta_a(1-\zeta_b)} \right), \quad (\text{B.98})$$

$$I_{\zeta_{2/3}} = \int_{\zeta_a}^{\zeta_b} d\zeta_{2/3} \frac{1}{1+\zeta_{2/3}} = \ln(1+\zeta_{2/3}) \Big|_{\zeta_a}^{\zeta_b} = \ln \left(\frac{1+\zeta_b}{1+\zeta_a} \right), \quad (\text{B.99})$$

$$I_{\zeta_4 1} = \int_{\zeta_a}^{\zeta_b} d\zeta_4 \zeta_4^{-1-\alpha} = \begin{cases} \frac{\zeta^{-\alpha}}{-\alpha} \Big|_{\zeta_a}^{\zeta_b} = \frac{\zeta_b^{-\alpha} - \zeta_a^{-\alpha}}{\alpha} & \text{for } \alpha \neq 0 \\ \ln \zeta \Big|_{\zeta_a}^{\zeta_b} = \ln \frac{\zeta_b}{\zeta_a} & \text{for } \alpha = 0 \end{cases} \quad (\text{B.100})$$

$$I_{\zeta_4 2} = \int_{\zeta_a}^{\zeta_b} d\zeta_4 \zeta_4^{-\alpha} = \begin{cases} \frac{\zeta^{1-\alpha}}{1-\alpha} \Big|_{\zeta_a}^{\zeta_b} = \frac{\zeta_b^{1-\alpha} - \zeta_a^{1-\alpha}}{1-\alpha} & \text{for } \alpha \neq 1 \\ \ln \zeta \Big|_{\zeta_a}^{\zeta_b} = \ln \frac{\zeta_b}{\zeta_a} & \text{for } \alpha = 1 \end{cases}. \quad (\text{B.101})$$

The trial value for ζ is found by inverting the equation

$$\mathcal{R}_\zeta = \frac{I_\zeta(\zeta_{\min}, \zeta)}{I_\zeta(\zeta_{\min}, \zeta_{\max})}, \quad (\text{B.102})$$

the solutions are

$$\zeta_1 = \left[1 + \frac{1-\zeta_{\min}}{\zeta_{\min}} \left(\frac{\zeta_{\min}(1-\zeta_{\max})}{\zeta_{\max}(1-\zeta_{\min})} \right)^{\mathcal{R}_{\zeta_1}} \right]^{-1}, \quad (\text{B.103})$$

$$\zeta_{2/3} = (1+\zeta_{\min}) \left(\frac{1+\zeta_{\max}}{1+\zeta_{\min}} \right)^{\mathcal{R}_{\zeta_{2/3}}} - 1, \quad (\text{B.104})$$

$$\zeta_4 = (\mathcal{R}_{\zeta_4 1}(\zeta_{\min}^{-\alpha} - \zeta_{\max}^{-\alpha}) + \zeta_{\max}^{-\alpha})^{-\frac{1}{\alpha}}, \quad (\text{B.105})$$

$$\zeta_4 = (\mathcal{R}_{\zeta_4 2}(\zeta_{\min}^{1-\alpha} - \zeta_{\max}^{1-\alpha}) + \zeta_{\max}^{1-\alpha})^{\frac{1}{1-\alpha}}. \quad (\text{B.106})$$

B.10 Generation of Trial Evolution Scale

The integral over the evolution is scale is

$$\int_{Q_{E \text{ new}}^2}^{Q_{EF}^2} \frac{dQ_E^2}{Q_E^2} = \ln Q_E^2 \Big|_{Q_{E \text{ new}}^2}^{Q_{EF}^2} = \ln \frac{Q_{EF}^2}{Q_{E \text{ new}}^2} \quad (\text{B.107})$$

The next trial scale is found by solving the equation

$$\hat{\Delta}(Q_E^2, Q_{E \text{ new}}^2) = \mathcal{R} \quad (\text{B.108})$$

for $Q_{E \text{ new}}^2$. For constant trial $\hat{\alpha}_s$, the solutions are

1. Soft with Q_\perp^2 and ζ_1 :

$$\mathcal{R} = \exp \left(-\frac{\hat{\alpha}_s \mathcal{C}}{2\pi} I_{\zeta_1} \hat{R}_f \ln \left(\frac{Q_\perp^2}{Q_{\perp \text{new}}^2} \right) \right) \Leftrightarrow Q_{\perp \text{new}}^2 = Q_\perp^2 \mathcal{R}^{\frac{2\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1} \hat{R}_f}} \quad (\text{B.109})$$

2. A gluon collinear with Q_\perp^2 and ζ_3 (similar B gluon collinear with ζ_2):

$$\mathcal{R} = \exp \left(-\frac{\hat{\alpha}_s \mathcal{C}}{2\pi} I_{\zeta_{2/3}} \hat{R}_{xf} \ln \left(\frac{Q_\perp^2}{Q_{\perp \text{new}}^2} \right) \right) \Leftrightarrow Q_{\perp \text{new}}^2 = Q_\perp^2 \mathcal{R}^{\frac{2\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_{2/3}} \hat{R}_{xf}}} \quad (\text{B.110})$$

3. A gluon splitting

(a) with Q_\perp^2 and ζ_3 (similar B gluon splitting with ζ_2):

$$\mathcal{R} = \exp \left(-\frac{\hat{\alpha}_s \mathcal{C}}{4\pi} I_{\zeta_{2/3}} \hat{R}_f \ln \left(\frac{Q_\perp^2}{Q_{\perp \text{new}}^2} \right) \right) \Leftrightarrow Q_{\perp \text{new}}^2 = Q_\perp^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1} \hat{R}_f}} \quad (\text{B.111})$$

(b) with Q_A^2 and ζ_4 (similar B gluon splitting with Q_B^2 and ζ_4):

$$\mathcal{R} = \exp \left(-\frac{\hat{\alpha}_s \mathcal{C}}{4\pi} I_{\zeta_{4\,1}} \hat{R}_{x^\alpha f} \ln \left(\frac{Q_A^2}{Q_{A \text{new}}^2} \right) \right) \Leftrightarrow Q_{A \text{new}}^2 = Q_A^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_{4\,1}} \hat{R}_{x^\alpha f}}} \quad (\text{B.112})$$

4. A conversion

(a) with Q_\perp^2 and ζ_3 (similar B conversion with ζ_2):

$$\mathcal{R} = \exp \left(-\frac{\hat{\alpha}_s \mathcal{C}}{4\pi} I_{\zeta_{2/3}} \hat{R}_{xf} \ln \left(\frac{Q_\perp^2}{Q_{\perp \text{new}}^2} \right) \right) \Leftrightarrow Q_{\perp \text{new}}^2 = Q_\perp^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_{2/3}} \hat{R}_{xf}}} \quad (\text{B.113})$$

(b) with Q_A^2 and ζ_4 (similar B conversion with Q_B^2 and ζ_4):

$$\mathcal{R} = \exp \left(-\frac{\hat{\alpha}_s \mathcal{C}}{4\pi} I_{\zeta_{4\,2}} \hat{R}_{x^\alpha f} \ln \left(\frac{Q_A^2}{Q_{A \text{new}}^2} \right) \right) \Leftrightarrow Q_{A \text{new}}^2 = Q_A^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_{4\,2}} \hat{R}_{x^\alpha f}}} \quad (\text{B.114})$$

Running of the Coupling: We use

$$\alpha_s(Q_E^2) = \frac{1}{b_0 \ln \left(\frac{k_R^2 Q_E^2}{\Lambda^2} \right)} \quad (\text{B.115})$$

to write the evolution kernels as

$$d\hat{\mathcal{A}} = \text{Com} \frac{dQ_E^2}{Q_E^2 \ln \left(\frac{k_R^2 Q_E^2}{\Lambda^2} \right)} \Leftrightarrow \hat{\mathcal{A}} = \text{Com} \ln \left(\frac{\ln \left(\frac{k_R^2 Q_E^2}{\Lambda^2} \right)}{\ln \left(\frac{k_R^2 Q_{E\text{new}}^2}{\Lambda^2} \right)} \right). \quad (\text{B.116})$$

The solutions for the next trial scale $Q_{E\text{new}}^2$ are therefore

$$\mathcal{R} = \exp \left(-\text{Com} \ln \left(\frac{\ln \left(\frac{k_R^2 Q_E^2}{\Lambda^2} \right)}{\ln \left(\frac{k_R^2 Q_{E\text{new}}^2}{\Lambda^2} \right)} \right) \right) \Leftrightarrow Q_{E\text{new}}^2 = \frac{\Lambda^2}{k_R^2} \left(\frac{k_R^2 Q_E^2}{\Lambda^2} \right)^{\mathcal{R}^{1/\text{Com}}}. \quad (\text{B.117})$$

To include two-loop running we use one-loop running as above and veto with

$$\frac{\alpha_s^{(2)}(Q_E^2, \Lambda_{\text{QCD}}^{(2)})}{\alpha_s^{(1)}(Q_E^2, \Lambda_{\text{QCD}}^{(2)})} \quad (\text{B.118})$$

B.11 Inverse Transforms

The inversions are for Q_\perp^2 and ζ_1

$$s_{jb} = \frac{Q_\perp^2}{\zeta_1} \quad (\text{B.119})$$

$$s_{aj} = \frac{Q_\perp^2 + \zeta_1 s_{AB}}{1 - \zeta_1} \quad (\text{B.120})$$

and for Q_\perp^2 and ζ_2

$$s_{aj} = \zeta_2 s_{AB} \quad (\text{B.121})$$

$$s_{jb} = \frac{Q_\perp^2 (1 + \zeta_2)}{\zeta_2 - Q_\perp^2 / s_{AB}} \quad (\text{B.122})$$

and for Q_\perp^2 and ζ_3

$$s_{jb} = \zeta_3 s_{AB} \quad (\text{B.123})$$

$$s_{aj} = \frac{Q_\perp^2 (1 + \zeta_3)}{\zeta_3 - Q_\perp^2 / s_{AB}}, \quad (\text{B.124})$$

and for Q_A^2 and ζ_4

$$s_{aj} = Q_A^2 \quad (\text{B.125})$$

$$s_{jb} = s_{AB}(\zeta_4 - 1) - Q_A^2, \quad (\text{B.126})$$

and for Q_B^2 and ζ_4

$$s_{jb} = Q_B^2 \quad (\text{B.127})$$

$$s_{aj} = s_{AB}(\zeta_4 - 1) - Q_B^2. \quad (\text{B.128})$$

C Initial–Final Evolution Equations

C.1 Notation and Kinematic Relations

We denote the pre- and post-branching partons by $AK \rightarrow ajk$, respectively. Conservation of energy and momentum implies $p_a + p_K = p_A + p_j + p_k$ or equivalently $p_a - p_j - p_k = p_A - p_K$. For general masses, with the notation $s_{ij} \equiv 2p_i \cdot p_j$, the relation between the pre-and post-branching invariants is thus

$$s_{AK} - m_A^2 - m_K^2 = s_{aj} + s_{ak} - s_{jk} - m_a^2 - m_j^2 - m_k^2. \quad (\text{C.1})$$

If all partons are massless, or if parton j is a gluon (gluon emission, such that $m_a = m_A$, $m_k = m_K$, and $m_j = 0$), the relation simplifies to

$$s_{AK} = s_{aj} + s_{ak} - s_{jk}. \quad (\text{C.2})$$

For gluon K splitting to massive quarks j and k , the relation for arbitrary $m_a = m_A$ and $m_j = m_k$ is

$$s_{AK} = s_{aj} + s_{ak} - s_{jk} - 2m_j^2. \quad (\text{C.3})$$

For a gluon A backwards-evolving to a quark a (gluon conversion), we have (irrespective of the recoiler mass $m_K = m_k$),

$$s_{AK} = s_{aj} + s_{ak} - s_{jk} - m_a^2 - m_j^2, \quad (\text{C.4})$$

where we leave the possibility open of defining the new incoming quark to have a virtuality different from that of the (on-shell) final-state one. For a quark A backwards-evolving to a gluon a (quark conversion), we have (irrespective of the recoiler mass $m_K = m_k$),

$$s_{AK} - m_A^2 = s_{aj} + s_{ak} - s_{jk} - m_j^2. \quad (\text{C.5})$$

which for the nominal case of $m_A = m_j$ reduces to the formula for the fully massless case given above.

For dimensionless equivalents, we normalise by the largest invariant, $2p_a \cdot (p_j + p_k) = s_{aj} + s_{ak}$, hence

$$y_{aj} = \frac{s_{aj}}{s_{aj} + s_{ak}}, \quad (\text{C.6})$$

$$y_{jk} = \frac{s_{jk}}{s_{aj} + s_{ak}}, \quad (\text{C.7})$$

$$y_{ak} = \frac{s_{ak}}{s_{aj} + s_{ak}}, \quad (\text{C.8})$$

where, for massless partons, the denominators can also be written $s_{aj} + s_{ak} = s_{AK} + s_{jk}$ and the dimensionless momentum-conservation relation implies

$$y_{AK} = 1 - y_{jk}, \quad (\text{C.9})$$

$$y_{ak} = 1 - y_{aj}, \quad (\text{C.10})$$

$$y_{aj} + y_{ak} = y_{AK} + y_{jk} = 1. \quad (\text{C.11})$$

C.2 Crossing Relations

Compared to the FF case, crossing symmetry for the case when recoiling parton i is crossed to be identified with a , implies

$$y_{ij} \rightarrow \frac{-s_{aj}}{-s_{AK}} \equiv \frac{y_{aj}}{y_{AK}} \quad (\text{C.12})$$

$$y_{jk} \rightarrow \frac{s_{jk}}{-s_{AK}} \equiv \frac{-y_{jk}}{y_{AK}} \quad (\text{C.13})$$

$$y_{ik} \rightarrow \frac{-s_{ak}}{-s_{AK}} \equiv \frac{y_{ak}}{y_{AK}}. \quad (\text{C.14})$$

Crossings for which parton k is identified with a and parton i becomes parton k (e.g., for the GQemitIF antenna function) have:

$$y_{ij} \rightarrow \frac{s_{jk}}{-s_{AK}} \equiv \frac{-y_{jk}}{y_{AK}} \quad (\text{C.15})$$

$$y_{jk} \rightarrow \frac{-s_{aj}}{-s_{AK}} \equiv \frac{y_{aj}}{y_{AK}} \quad (\text{C.16})$$

$$y_{ik} \rightarrow \frac{-s_{ak}}{-s_{AK}} \equiv \frac{y_{ak}}{y_{AK}}. \quad (\text{C.17})$$

Note also that outgoing particles are mapped to incoming particles with the opposite helicities, so that, e.g., the IF antenna function for $(\bar{q}_A^+ \bar{q}_K^+ \rightarrow \bar{q}_a^+ g_j^+ \bar{q}_k^+)$ is obtained from the FF one for $(q_I^- \bar{q}_K^+ \rightarrow q_i^- g_j^+ \bar{q}_k^+)$.

Finally, as for the case of the II functions, note that the initial-state parts of VINCIA's antenna functions are 'sectorised' since there is no sector for 'emission into the initial state'. The initial-state functions with gluon parents are therefore mostly not simple crossings of the corresponding final-state ones. The IF case is a hybrid, with the initial-state legs being sectorised, while the final-state legs are global.

C.3 Antenna Functions

C.3.1 QQemitIF

The helicity average (for unpolarised partons) is:

$$a(q_A q_K \rightarrow q_a g_j q_k) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{aj})^2 + (1 - y_{jk})^2}{y_{aj} y_{jk}} + \frac{3}{2} - \frac{y_{aj}^2}{2} - \frac{y_{jk}^2}{2} \right], \quad (\text{C.18})$$

The individual helicity contributions (for massless partons with specified helicities) are chosen such that all antenna functions remain positive definite over all of phase space.

$$a(++ \rightarrow +++) = \frac{1}{s_{AK}} \left[\frac{1}{y_{aj} y_{jk}} \right], \quad (\text{C.19})$$

$$a(++ \rightarrow +-+) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{aj})^2 + (1 - y_{jk})^2 - 1}{y_{aj} y_{jk}} + 3 - y_{aj}^2 - y_{jk}^2 \right], \quad (\text{C.20})$$

$$a(+- \rightarrow +++-) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{aj})^2}{y_{aj} y_{jk}} \right], \quad (\text{C.21})$$

$$a(+- \rightarrow +--) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{jk})^2}{y_{aj} y_{jk}} \right]. \quad (\text{C.22})$$

Note 1: the sum of the two ++ antenna functions has the same singularities as the sum of the two +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

Note 2: the direct crossing of the second (++) antenna has a zero across the diagonal of the IF phase space, corresponding to points with $s_{ak} = s_{jk}$ or equivalently $s_{AK} = s_{aj}$. The former condition is satisfied when $(p_a - p_j) \cdot p_k \rightarrow 0$. Not having come up with a good physical reason why that antenna function should go to zero there, we have chosen to add nonsingular terms as shown above.

C.3.2 QGemitIF

These functions are identical to their FF counterparts, with the replacements $y_{ij} \rightarrow y_{aj}$ and $y_{ik} \rightarrow y_{AK}$.

The helicity average (for unpolarised partons) is:

$$a(q_A g_K \rightarrow q_a g_j g_k) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{aj})^3 + (1 - y_{jk})^2}{y_{aj} y_{jk}} + \frac{1 - 2y_{aj}}{y_{jk}} + \frac{3}{2} + y_{aj} - \frac{y_{jk}}{2} - \frac{y_{aj}^2}{2} \right]. \quad (\text{C.23})$$

The individual helicity contributions (for massless partons with specified helicities) are:

$$a(++ \rightarrow +++) = \frac{1}{s_{AK}} \left[\frac{1}{y_{aj}y_{jk}} + \frac{1-2y_{aj}}{y_{jk}} \right], \quad (C.24)$$

$$a(++ \rightarrow +-+) = \frac{1}{s_{AK}} \left[\frac{(1-y_{aj})^3 + (1-y_{jk})^2 - 1}{y_{aj}y_{jk}} + 3 - y_{aj}^2 \right], \quad (C.25)$$

$$a(+ - \rightarrow +++) = \frac{1}{s_{AK}} \left[\frac{(1-y_{aj})^3}{y_{aj}y_{jk}} \right], \quad (C.26)$$

$$a(+ - \rightarrow +- -) = \frac{1}{s_{AK}} \left[\frac{(1-y_{jk})^2}{y_{aj}y_{jk}} + \frac{1-2y_{aj}}{y_{jk}} + 2y_{aj} - y_{jk} \right]. \quad (C.27)$$

Note that the sum of the two ++ antenna functions has the same singularities as the sum of the two +- ones, thus the same singular terms are obtained when summing over the helicity of the emitted gluon, irrespective of parent helicities.

Note 2: the α collinear partitioning parameter is not written explicitly here. It belongs on the antisymmetric term, with $1 - 2y_{aj} = y_{ak} - y_{aj}$. Whether or not to include the nonsingular terms is arbitrary. They are required here to ensure positivity for the default $\alpha = 0$.

Note 3: the nonsingular terms are to ensure positive-definite functions which do not vanish at the arbitrary line across the diagonal of the phase space while still vanishing for hard-collinear helicity flips.

C.3.3 GQemitIF

The helicity average (for unpolarised partons) is:

$$a(g_A q_K \rightarrow g_a g_j q_k) = \frac{1}{s_{AK}} \left[\frac{(1-y_{jk})^3 + (1-y_{aj})^2}{y_{aj}y_{jk}} + \frac{1+y_{jk}^3}{y_{aj}(y_{AK}+y_{aj})} + \frac{3}{2} - \frac{y_{jk}^2}{2} \right]. \quad (C.28)$$

Since the gluon is in the initial state, the individual helicity contributions (for massless partons with specified helicities) are crossings of corresponding sector FF functions, up to non-singular terms.

$$a(++ \rightarrow +++) = \frac{1}{s_{AK}} \left[\frac{1}{y_{aj}y_{jk}} + \frac{1}{y_{aj}(y_{AK}+y_{aj})} \right], \quad (C.29)$$

$$a(++ \rightarrow +-+) = \frac{1}{s_{AK}} \left[\frac{(1-y_{aj})^2 + (1-y_{jk})^3 - 1}{y_{aj}y_{jk}} + 3 - y_{jk}^2 \right], \quad (C.30)$$

$$a(+ - \rightarrow +++) = \frac{1}{s_{AK}} \left[\frac{(1-y_{aj})^2}{y_{aj}y_{jk}} + \frac{1}{y_{aj}(y_{AK}+y_{aj})} \right], \quad (C.31)$$

$$a(+ - \rightarrow +- -) = \frac{1}{s_{AK}} \left[\frac{(1-y_{jk})^3}{y_{aj}y_{jk}} \right]. \quad (C.32)$$

Note: the nonsingular terms for the helicity-flip antenna are chosen such that the function still goes to zero in the hard-collinear limits but allows it to be non-zero in the hard part of phase space.

The two additional antennae, with helicity flips on the incoming gluon leg (i.e., with parton j inheriting the a helicity, rather than A) are:

$$a(++ \rightarrow --+) = \frac{1}{s_{AK}} \frac{y_{jk}^3}{y_{aj}(y_{AK} + y_{aj})}, \quad (\text{C.33})$$

$$a(+- \rightarrow ---) = a(++ \rightarrow --+). \quad (\text{C.34})$$

C.3.4 GGemitIF

The GGemitIF functions are essentially hybrids between sector antenna functions for the initial-state singularities and global ones for the final-state legs. The helicity average (for unpolarised partons) is:

$$a(g_A g_K \rightarrow g_a g_j q_k) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{aj})^3 + (1 - y_{jk})^3}{y_{aj} y_{jk}} + \frac{1 + y_{jk}^3}{y_{aj}(y_{AK} + y_{aj})} + \frac{1 - 2y_{aj}}{y_{jk}} + 3 - 2y_{jk} \right]. \quad (\text{C.35})$$

The helicity contributions are:

$$a(++ \rightarrow +++) = \frac{1}{s_{AK}} \left[\frac{1}{y_{aj} y_{jk}} + \frac{1 - 2y_{aj}}{y_{jk}} + \frac{1}{y_{aj}(y_{AK} + y_{aj})} \right], \quad (\text{C.36})$$

$$a(++ \rightarrow +-+) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{aj})^3 + (1 - y_{jk})^3 - 1}{y_{aj} y_{jk}} + 6 - 3y_{aj} - 3y_{jk} + y_{aj} y_{jk} \right], \quad (\text{C.37})$$

$$a(+- \rightarrow ++-) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{aj})^3}{y_{aj} y_{jk}} + \frac{1}{y_{aj}(y_{AK} + y_{aj})} \right], \quad (\text{C.38})$$

$$a(+- \rightarrow +- -) = \frac{1}{s_{AK}} \left[\frac{(1 - y_{jk})^3}{y_{aj} y_{jk}} + \frac{1 - 2y_{aj}}{y_{jk}} + 3y_{aj} - y_{jk} - y_{aj} y_{jk} \right]. \quad (\text{C.39})$$

Note: the nonsingular terms for the helicity-flip antenna are chosen such that it remains positive-definite over the phase space with $p_{\perp}^2 < s_{AK}$ and still goes to zero in the hard-collinear limits.

Note 2: the last function generates some quadratic terms to remain positive definite.

The two additional antennae, with helicity flips on the incoming gluon leg (i.e., with parton j inheriting the a helicity, rather than A) are the same as for the GQemitIF case:

$$a(++ \rightarrow --+) = \frac{1}{s_{AK}} \frac{y_{jk}^3}{y_{aj}(y_{AK} + y_{aj})}, \quad (\text{C.40})$$

$$a(+- \rightarrow ---) = a(++ \rightarrow --+). \quad (\text{C.41})$$

C.3.5 XGsplitIF

The XGsplitIF functions are essentially identical to the final-state gluon-splitting antennae, the only difference being that the recoiler is now an initial-state parton. The helicity average (for unpolarised partons, including an optional correction term for splitting to massive quarks) is:

$$a(X_{AGK} \rightarrow X_a \bar{q}_j q_k) = \frac{1}{2m_{jk}^2} \left[y_{ak}^2 + y_{aj}^2 + \frac{2m_j^2}{m_{jk}^2} \right]. \quad (\text{C.42})$$

The helicity contributions are:

$$a(X+ \rightarrow X-+) = \frac{y_{ak}^2}{2m_{jk}^2}, \quad (\text{C.43})$$

$$a(X+ \rightarrow X+-) = \frac{y_{aj}^2}{2m_{jk}^2}. \quad (\text{C.44})$$

Note, in principle the exact crossing corresponds to performing the substitution $m_{jk}^2 \rightarrow m_{jk}^2 y_{AK}^2$ in the denominator above, which is neglected since the $y_{jk} \rightarrow 0$ limit corresponds to $y_{AK} \rightarrow 1$.

C.3.6 QXsplitIF

The QXsplitIF functions are essentially identical to the QXsplitII functions, the only difference being that the recoiler is now a final-state parton. STILL TENTATIVE, NOT CHECKED VIA EXPLICIT CROSSING YET.

$$a(q_A X_K \rightarrow g_a \bar{q}_j X_k) = \frac{1}{s_{AK}} \frac{y_{AK}^2 + (1 - y_{AK})^2}{y_{aj}} \quad (\text{C.45})$$

The individual helicity contributions are:

$$a(+X \rightarrow +-X) = \frac{1}{s_{AK}} \frac{y_{AK}^2}{y_{aj}} \quad (\text{C.46})$$

$$a(+X \rightarrow --X) = \frac{1}{s_{AK}} \frac{(1 - y_{AK})^2}{y_{aj}} \quad (\text{C.47})$$

$$a(-X \rightarrow -+X) = \frac{1}{s_{AK}} \frac{y_{AK}^2}{y_{aj}} \quad (\text{C.48})$$

$$a(-X \rightarrow ++X) = \frac{1}{s_{AK}} \frac{(1 - y_{AK})^2}{y_{aj}} \quad (\text{C.49})$$

$$(\text{C.50})$$

C.3.7 GXconvIF

The GXconvIF functions are essentially identical to the GXconvII functions, the only difference being that the recoiler is now a final-state parton. STILL TENTATIVE, NOT CHECKED VIA EXPLICIT CROSSING YET.

The helicity average (for unpolarised partons) is

$$a(g_A X_K \rightarrow q_a q_j X_k) = \frac{1}{s_{AK}} \frac{1 + (1 - y_{AK})^2}{2y_{aj}(y_{AK} + y_{aj})} . \quad (\text{C.51})$$

The individual helicity contributions are:

$$a(+X \rightarrow ++X) = \frac{1}{s_{AK}} \frac{1}{2y_{aj}(y_{AK} + y_{aj})} \quad (\text{C.52})$$

$$a(+X \rightarrow --X) = \frac{1}{s_{AK}} \frac{(1 - y_{AK})^2}{2y_{aj}(y_{AK} + y_{aj})} \quad (\text{C.53})$$

$$a(-X \rightarrow --X) = \frac{1}{s_{AK}} \frac{1}{2y_{aj}(y_{AK} + y_{aj})} \quad (\text{C.54})$$

$$a(-X \rightarrow ++X) = \frac{1}{s_{AK}} \frac{(1 - y_{AK})^2}{2y_{aj}(y_{AK} + y_{aj})} \quad (\text{C.55})$$

$$(\text{C.56})$$

C.4 Evolution Variables

The evolution variables we use are

$$Q_{\perp}^2 = \frac{s_{aj}s_{jk}}{s_{AK} + s_{jk}} , \quad (\text{C.57})$$

$$Q_A^2 = s_{aj} , \quad (\text{C.58})$$

$$Q_K^2 = s_{jk} . \quad (\text{C.59})$$

Phase Space Boundaries: With $s_{AK} + s_{jk} = s_{aj} + s_{ak}$ and $s_{jk} = -s_{AK} + s_{aK} = (x_a - x_A)\sqrt{s}p_{K-}$, the phase space boundaries are $0 \leq s_{jk} \leq \frac{1-x_A}{x_A}s_{AK}$ and $0 \leq s_{aj} \leq s_{AK} + s_{jk}$. The maxima of the evolution variables are

$$Q_{\perp \max}^2 = \frac{1 - x_A}{x_A} s_{AK} , \quad (\text{C.60})$$

$$Q_{A \max}^2 = s_{AK}/x_A , \quad (\text{C.61})$$

$$Q_{K \max}^2 = \frac{1 - x_A}{x_A} s_{AK} . \quad (\text{C.62})$$

C.5 Zeta Definitions

The choices are

$$\zeta_1 = \frac{s_{jk} + s_{AK}}{s_{AK}} = \frac{x_a}{x_A}, \quad (\text{C.63})$$

$$\zeta_2 = \frac{s_{aj}}{s_{AK} + s_{jk}}, \quad (\text{C.64})$$

$$\zeta_3 = \frac{s_{aj}}{s_{jk}}. \quad (\text{C.65})$$

The integration boundaries for the ζ variables are

$$\zeta_{1-}(Q_\perp^2) = \frac{Q_\perp^2 + s_{AK}}{s_{AK}} \quad \zeta_{1+}(Q_\perp^2) = \frac{1}{x_A}, \quad (\text{C.66})$$

$$\zeta_{2-}(Q_\perp^2) = \frac{Q_\perp^2 x_A}{s_{AK}(1 - x_A)} \quad \zeta_{2+}(Q_\perp^2) = 1, \quad (\text{C.67})$$

$$\zeta_{3-}(Q_\perp^2) = \frac{Q_\perp^2 x_A}{s_{AK}(1 - x_A)^2} \quad \zeta_{3+}(Q_\perp^2) = \frac{s_{AK} + Q_\perp^2}{Q_\perp^2}, \quad (\text{C.68})$$

$$\zeta_{1-}(Q_A^2) = \max\left(1, \frac{Q_A^2}{s_{AK}}\right) \quad \zeta_{1+}(Q_A^2) = \frac{1}{x_A}, \quad (\text{C.69})$$

$$\zeta_{2-}(Q_K^2) = 0 \quad \zeta_{2+}(Q_K^2) = 1. \quad (\text{C.70})$$

C.6 Jacobians

The Jacobians for the transformation from the phase-space variables, (s_{aj}, s_{jk}) , to the shower variables, (Q_E, ζ) , are

$$|J(Q_\perp^2, \zeta_1)| = \frac{(s_{AK} + s_{jk})s_{AK}}{s_{jk}}, \quad (\text{C.71})$$

$$|J(Q_\perp^2, \zeta_2)| = \frac{(s_{AK} + s_{jk})^2}{s_{aj}}, \quad (\text{C.72})$$

$$|J(Q_\perp^2, \zeta_3)| = \frac{1}{\zeta_3} \frac{(s_{AK} + s_{jk})^2}{(2s_{AK} + s_{jk})}, \quad (\text{C.73})$$

$$|J(Q_A^2, \zeta_1)| = s_{AK}, \quad (\text{C.74})$$

$$|J(Q_K^2, \zeta_2)| = s_{AK} + s_{jk}. \quad (\text{C.75})$$

C.7 Trial Functions

The following trial functions are used:

$$a_{\text{soft}} = 2 \frac{s_{ak}}{s_{aj} s_{jk}} \Rightarrow \hat{a}_{\text{soft}} = \frac{2s_{AK} + s_{jk}}{Q_{\perp}^2 s_{AK}}, \quad (\text{C.76})$$

$$a_{\text{q coll A}} = \frac{1}{s_{AK}} \frac{s_{jk}}{s_{aj}} \Rightarrow \hat{a}_{\text{q coll A}} = \frac{1}{s_{AK}} \frac{s_{jk} + s_{AK}}{s_{aj}}, \quad (\text{C.77})$$

$$a_{\text{q coll K}} = \frac{1}{s_{AK}} \frac{s_{aj}}{s_{jk}} \Rightarrow \hat{a}_{\text{q coll K}} = \frac{1}{s_{AK}} \frac{s_{jk} + s_{AK}}{s_{jk}}, \quad (\text{C.78})$$

$$a_{\text{g coll1 A}} = \frac{2}{s_{AK}} \frac{s_{jk}}{s_{aj}} \frac{s_{ak}}{s_{AK}} \Rightarrow \hat{a}_{\text{g coll1 A}} = 2 \frac{(s_{AK} + s_{jk})^2}{s_{AK}^2 s_{aj}}, \quad (\text{C.79})$$

$$a_{\text{g coll2 A}} = \frac{2s_{jk}}{s_{aj}(s_{AK} + s_{jk})} \Rightarrow \hat{a}_{\text{g coll2 A}} = \frac{2}{s_{aj}}, \quad (\text{C.80})$$

$$a_{\text{g coll K}} = \frac{1}{s_{AK}} \frac{s_{aj}}{s_{jk}} \frac{s_{ak}}{s_{AK}} \Rightarrow \hat{a}_{\text{g coll K}} = \frac{1}{s_{AK}} \frac{s_{aj}}{s_{jk}} \frac{s_{AK} + s_{jk}}{s_{AK}}, \quad (\text{C.81})$$

$$a_{\text{conv A}} = \frac{1}{2} \frac{1}{s_{aj}} \frac{s_{ak}^2 + s_{jk}^2}{s_{AK}^2} \Rightarrow \hat{a}_{\text{conv A}} = \frac{1}{s_{aj}} \frac{(s_{AK} + s_{jk})^2}{s_{AK}^2}, \quad (\text{C.82})$$

$$a_{\text{split A}} = \frac{1}{s_{AK}} \frac{s_{ak}}{s_{aj}} - \frac{2}{s_{AK}} \frac{s_{jk}}{s_{aj}} \frac{s_{AK} - s_{aj}}{s_{AK} + s_{jk}} \Rightarrow \hat{a}_{\text{split A}} = \frac{2}{s_{AK}} \frac{s_{AK} + s_{jk}}{s_{aj}}, \quad (\text{C.83})$$

$$a_{\text{split K}} = \frac{1}{2} \frac{1}{s_{jk}} \frac{s_{ak}^2 + s_{aj}^2}{s_{AK}^2} \Rightarrow \hat{a}_{\text{split K}} = \frac{1}{2} \frac{1}{s_{jk}} \frac{(s_{AK} + s_{jk})^2}{s_{AK}^2}. \quad (\text{C.84})$$

C.8 Integration Kernels

The overestimate of the evolution integral is

$$\hat{A}(Q_{E\text{F}}^2, Q_{E\text{new}}^2) = \int_{Q_{E\text{new}}^2}^{Q_{E\text{F}}^2} \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \hat{a} \hat{R}_f |J| \text{d}Q_E^2 \text{d}\zeta \frac{\text{d}\phi}{2\pi}. \quad (\text{C.85})$$

The integration kernels for the Q_E^2 integration are

1. Soft with Q_{\perp}^2 and ζ_3 :

$$\begin{aligned} \text{d}\hat{\mathcal{A}}_{\text{soft}}(Q_{\perp}^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{2s_{AK} + s_{jk}}{Q_{\perp}^2 s_{AK}} \frac{1}{\zeta_3} \frac{(s_{AK} + s_{jk})^2}{(2s_{AK} + s_{jk})} \hat{R}_f \text{d}Q_{\perp}^2 \text{d}\zeta_3 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{\text{d}Q_{\perp}^2}{Q_{\perp}^2} \frac{\text{d}\zeta_3}{\zeta_3} \end{aligned} \quad (\text{C.86})$$

2. A quark collinear with Q_\perp^2 and ζ_1 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{q coll A}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{1}{s_{AK}} \frac{s_{jk} + s_{AK}}{s_{aj}} \frac{(s_{AK} + s_{jk})s_{AK}}{s_{jk}} \hat{R}_f dQ_\perp^2 d\zeta_1 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{xf} \frac{dQ_\perp^2}{Q_\perp^2} d\zeta_1 \frac{x_A}{x_a} \left(\frac{x_a}{x_A} \frac{1}{\zeta_1} \right)^{1+\beta} \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{x^\beta f} \frac{dQ_\perp^2}{Q_\perp^2} \frac{d\zeta_1}{\zeta_1^{1+\beta}}
\end{aligned} \tag{C.87}$$

3. K quark collinear with Q_\perp^2 and ζ_2 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{q coll K}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{1}{s_{AK}} \frac{s_{jk} + s_{AK}}{s_{jk}} \frac{(s_{AK} + s_{jk})^2}{s_{aj}} \hat{R}_f dQ_\perp^2 d\zeta_2 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{dQ_\perp^2}{Q_\perp^2} d\zeta_2
\end{aligned} \tag{C.88}$$

4. A gluon collinear 1 with Q_\perp^2 and ζ_1 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{g coll 1 A}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} 2 \frac{(s_{AK} + s_{jk})^2}{s_{AK}^2 s_{aj}} \frac{(s_{AK} + s_{jk})s_{AK}}{s_{jk}} \hat{R}_f dQ_\perp^2 d\zeta_1 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{dQ_\perp^2}{Q_\perp^2} d\zeta_1 \left(\frac{x_a}{x_A} \frac{1}{\zeta_1} \right)^\beta \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_{x^\beta f} \frac{dQ_\perp^2}{Q_\perp^2} \frac{d\zeta_1}{\zeta_1^\beta}
\end{aligned} \tag{C.89}$$

5. A gluon collinear 2 with Q_\perp^2 and ζ_1 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{g coll 2 A}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{2}{s_{aj}} \frac{(s_{AK} + s_{jk})s_{AK}}{s_{jk}} \hat{R}_f dQ_\perp^2 d\zeta_1 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{dQ_\perp^2}{Q_\perp^2} d\zeta_1 \left(\frac{x_a}{x_A} \right)^2 \left(\frac{x_a}{x_A} \frac{1}{\zeta_1} \right)^{2+\beta} \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_{x^\beta f} \frac{dQ_\perp^2}{Q_\perp^2} \frac{d\zeta_1}{\zeta_1^{2+\beta}}
\end{aligned} \tag{C.90}$$

6. K gluon collinear with Q_\perp^2 and ζ_2 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{g coll K}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{1}{s_{AK}} \frac{s_{aj}}{s_{jk}} \frac{s_{AK} + s_{jk}}{s_{AK}} \frac{(s_{AK} + s_{jk})^2}{s_{aj}} \hat{R}_f dQ_\perp^2 d\zeta_2 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{xf} \frac{dQ_\perp^2}{Q_\perp^2} \zeta_2 d\zeta_2
\end{aligned} \tag{C.91}$$

7. A conversion

(a) with Q_\perp^2 and ζ_1 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{conv A}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{1}{s_{aj}} \frac{(s_{AK} + s_{jk})^2}{s_{AK}^2} \frac{(s_{AK} + s_{jk})s_{AK}}{s_{jk}} \hat{R}_f dQ_\perp^2 d\zeta_1 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{dQ_\perp^2}{Q_\perp^2} d\zeta_1 \left(\frac{x_a}{x_A} \frac{1}{\zeta_1} \right)^\beta \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{x^\beta f} \frac{dQ_\perp^2}{Q_\perp^2} \frac{d\zeta_1}{\zeta_1^\beta}
\end{aligned} \tag{C.92}$$

(b) with Q_A^2 and ζ_1 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{conv A}}(Q_A^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{1}{s_{aj}} \frac{(s_{AK} + s_{jk})^2}{s_{AK}^2} s_{AK} \hat{R}_f dQ_\perp^2 d\zeta_1 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_f \frac{dQ_A^2}{Q_A^2} d\zeta_1 \left(\frac{x_a}{x_A} \frac{1}{\zeta_1} \right)^\gamma \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{x^\gamma f} \frac{dQ_A^2}{Q_A^2} \frac{d\zeta_1}{\zeta_1^\gamma}
\end{aligned} \tag{C.93}$$

8. A gluon splitting

(a) with Q_\perp^2 and ζ_1 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{split A}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{2}{s_{AK}} \frac{s_{AK} + s_{jk}}{s_{aj}} \frac{(s_{AK} + s_{jk})s_{AK}}{s_{jk}} \hat{R}_f dQ_\perp^2 d\zeta_1 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{dQ_\perp^2}{Q_\perp^2} d\zeta_1 \frac{x_A}{x_a} \left(\frac{x_a}{x_A} \frac{1}{\zeta_1} \right)^{1+\beta} \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \hat{R}_{x^\beta f} \frac{dQ_\perp^2}{Q_\perp^2} \frac{d\zeta_1}{\zeta_1^{1+\beta}}
\end{aligned} \tag{C.94}$$

(b) with Q_A^2 and ζ_1 :

$$\begin{aligned}
d\hat{\mathcal{A}}_{\text{split A}}(Q_A^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{2}{s_{AK}} \frac{s_{AK} + s_{jk}}{s_{aj}} s_{AK} \hat{R}_f dQ_A^2 d\zeta_1 \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_f \frac{dQ_A^2}{Q_A^2} d\zeta_1 \frac{x_A}{x_a} \left(\frac{x_a}{x_A} \frac{1}{\zeta_1} \right)^{1+\gamma} \\
&= \frac{\hat{\alpha}_s \mathcal{C}}{2\pi} \hat{R}_{x^\gamma f} \frac{dQ_A^2}{Q_A^2} \frac{d\zeta_1}{\zeta_1^{1+\gamma}}
\end{aligned} \tag{C.95}$$

9. K gluon splitting

(a) with Q_\perp^2 and ζ_2 :

$$\begin{aligned} d\hat{\mathcal{A}}_{\text{split K}}(Q_\perp^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{1}{2} \frac{1}{s_{jk}} \frac{(s_{AK} + s_{jk})^2}{s_{AK}^2} \frac{(s_{AK} + s_{jk})^2}{s_{aj}} \hat{R}_f dQ_\perp^2 d\zeta_2 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{8\pi} \hat{R}_{xf} \frac{dQ_\perp^2}{Q_\perp^2} d\zeta_2 \end{aligned} \quad (\text{C.96})$$

(b) with Q_K^2 and ζ_2 :

$$\begin{aligned} d\hat{\mathcal{A}}_{\text{split K}}(Q_K^2) &= \frac{\hat{\alpha}_s \mathcal{C}}{4\pi} \frac{s_{AK}}{(s_{AK} + s_{jk})^2} \frac{1}{2} \frac{1}{s_{jk}} \frac{(s_{AK} + s_{jk})^2}{s_{AK}^2} (s_{AK} + s_{jk}) \hat{R}_f dQ_K^2 d\zeta_2 \\ &= \frac{\hat{\alpha}_s \mathcal{C}}{8\pi} \hat{R}_{xf} \frac{dQ_K^2}{Q_K^2} d\zeta_2 \end{aligned} \quad (\text{C.97})$$

Note that

$$\hat{R}_{xf} = \frac{x_a}{x_A} \hat{R}_f = \frac{s_{AK} + s_{jk}}{s_{AK}} \hat{R}_f \quad (\text{C.98})$$

and that we introduced general factors β and γ to control the PDF ratio better.

C.9 Zeta Integrals and Generation of Trial Zeta

We have five different ζ integrals to solve,

$$I_{\zeta, \text{lin}} = \int_{\zeta_a}^{\zeta_b} d\zeta = \zeta|_{\zeta_a}^{\zeta_b} = \zeta_b - \zeta_a, \quad (\text{C.99})$$

$$I_{\zeta, \text{quad}} = \int_{\zeta_a}^{\zeta_b} \zeta d\zeta = \frac{\zeta^2}{2} \Big|_{\zeta_a}^{\zeta_b} = \frac{\zeta_b^2 - \zeta_a^2}{2}, \quad (\text{C.100})$$

$$I_{\zeta, \text{log}} = \int_{\zeta_a}^{\zeta_b} \frac{d\zeta}{\zeta} = \ln \zeta|_{\zeta_a}^{\zeta_b} = \ln \frac{\zeta_b}{\zeta_a}, \quad (\text{C.101})$$

$$I_{\zeta, \beta 1} = \int_{\zeta_a}^{\zeta_b} \frac{d\zeta}{\zeta^\beta} = \begin{cases} \frac{\zeta^{1-\beta}}{1-\beta} \Big|_{\zeta_a}^{\zeta_b} = \frac{\zeta_b^{1-\beta} - \zeta_a^{1-\beta}}{1-\beta} & \text{for } \beta \neq 1 \\ \ln \zeta|_{\zeta_a}^{\zeta_b} = \ln \frac{\zeta_b}{\zeta_a} & \text{for } \beta = 1 \end{cases} \quad (\text{C.102})$$

$$I_{\zeta, \beta 2} = \int_{\zeta_a}^{\zeta_b} \frac{d\zeta}{\zeta^{1+\beta}} = \begin{cases} \frac{\zeta^{-\beta}}{-\beta} \Big|_{\zeta_a}^{\zeta_b} = \frac{\zeta_a^{-\beta} - \zeta_b^{-\beta}}{-\beta} & \text{for } \beta \neq 0 \\ \ln \zeta|_{\zeta_a}^{\zeta_b} = \ln \frac{\zeta_b}{\zeta_a} & \text{for } \beta = 0 \end{cases} \quad (\text{C.103})$$

$$I_{\zeta, \beta 3} = \int_{\zeta_a}^{\zeta_b} \frac{d\zeta}{\zeta^{2+\beta}} = \begin{cases} \frac{\zeta^{-1-\beta}}{-1-\beta} \Big|_{\zeta_a}^{\zeta_b} = \frac{\zeta_a^{-1-\beta} - \zeta_b^{-1-\beta}}{-1-\beta} & \text{for } \beta \neq -1 \\ \ln \zeta|_{\zeta_a}^{\zeta_b} = \ln \frac{\zeta_b}{\zeta_a} & \text{for } \beta = -1 \end{cases}. \quad (\text{C.104})$$

The trial value for ζ is found by inverting the equation

$$\mathcal{R}_\zeta = \frac{I_\zeta(\zeta_{\min}, \zeta)}{I_\zeta(\zeta_{\min}, \zeta_{\max})}, \quad (\text{C.105})$$

the solutions are

$$\zeta_{\text{lin}} = \mathcal{R}_{\zeta_{\text{lin}}}(\zeta_{\text{min}} - \zeta_{\text{max}}) + \zeta_{\text{max}} , \quad (\text{C.106})$$

$$\zeta_{\text{quad}} = \sqrt{\mathcal{R}_{\zeta_{\text{quad}}}(\zeta_{\text{min}}^2 - \zeta_{\text{max}}^2) + \zeta_{\text{max}}^2} , \quad (\text{C.107})$$

$$\zeta_{\text{log}} = \zeta_{\text{max}} \left(\frac{\zeta_{\text{min}}}{\zeta_{\text{max}}} \right)^{\mathcal{R}_{\zeta_{\text{log}}}} , \quad (\text{C.108})$$

$$\zeta_{\beta 1} = \left(\mathcal{R}_{\zeta_{\beta 1}}(\zeta_{\text{min}}^{1-\beta} - \zeta_{\text{max}}^{1-\beta}) + \zeta_{\text{max}}^{1-\beta} \right)^{\frac{1}{1-\beta}} , \quad (\text{C.109})$$

$$\zeta_{\beta 2} = \left(\mathcal{R}_{\zeta_{\beta 2}}(\zeta_{\text{min}}^{-\beta} - \zeta_{\text{max}}^{-\beta}) + \zeta_{\text{max}}^{-\beta} \right)^{-\frac{1}{\beta}} , \quad (\text{C.110})$$

$$\zeta_{\beta 3} = \left(\mathcal{R}_{\zeta_{\beta 3}}(\zeta_{\text{min}}^{-1-\beta} - \zeta_{\text{max}}^{-1-\beta}) + \zeta_{\text{max}}^{-1-\beta} \right)^{-\frac{1}{-1-\beta}} . \quad (\text{C.111})$$

C.10 Generation of Trial Evolution Scale

The integral over the evolution is scale is

$$\int_{Q_{E \text{ new}}^2}^{Q_{E \text{ F}}^2} \frac{dQ_E^2}{Q_E^2} = \ln Q_E^2 \Big|_{Q_{E \text{ new}}^2}^{Q_{E \text{ F}}^2} = \ln \frac{Q_{E \text{ F}}^2}{Q_{E \text{ new}}^2} . \quad (\text{C.112})$$

The next trial scale is found by solving the equation

$$\hat{\Delta}(Q_E^2, Q_{E \text{ new}}^2) = \mathcal{R} \quad (\text{C.113})$$

for $Q_{E \text{ new}}^2$. For constant trial $\hat{\alpha}_s$, the solutions are

1. Soft with Q_{\perp}^2 and ζ_3 :

$$Q_{\perp \text{ new}}^2 = Q_{\perp}^2 \mathcal{R}^{\hat{\alpha}_s \mathcal{C}} \frac{4\pi}{I_{\zeta_3, \log}} \frac{1}{\hat{R}_f} \quad (\text{C.114})$$

2. A quark collinear with Q_{\perp}^2 and ζ_1 :

$$Q_{\perp \text{ new}}^2 = Q_{\perp}^2 \mathcal{R}^{\hat{\alpha}_s \mathcal{C}} \frac{4\pi}{I_{\zeta_1, \beta 2}} \frac{1}{\hat{R}_{x\beta f}} \quad (\text{C.115})$$

3. K quark collinear with Q_{\perp}^2 and ζ_2 :

$$Q_{\perp \text{ new}}^2 = Q_{\perp}^2 \mathcal{R}^{\hat{\alpha}_s \mathcal{C}} \frac{4\pi}{I_{\zeta_1, \text{lin}}} \frac{1}{\hat{R}_f} \quad (\text{C.116})$$

4. A gluon collinear 1 with Q_\perp^2 and ζ_1 :

$$Q_{\perp\text{new}}^2 = Q_\perp^2 \mathcal{R} \frac{2\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1, \beta 1} \hat{R}_{x^\beta f}} \quad (\text{C.117})$$

5. A gluon collinear 2 with Q_\perp^2 and ζ_1 :

$$Q_{\perp\text{new}}^2 = Q_\perp^2 \mathcal{R} \frac{4\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1, \beta 3} \hat{R}_{x^\beta f}} \quad (\text{C.118})$$

6. K gluon collinear with Q_\perp^2 and ζ_2 :

$$Q_{\perp\text{new}}^2 = Q_\perp^2 \mathcal{R} \frac{4\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_2, \text{quad}} \hat{R}_{xf}} \quad (\text{C.119})$$

7. A conversion

(a) with Q_\perp^2 and ζ_1 :

$$Q_{\perp\text{new}}^2 = Q_\perp^2 \mathcal{R} \frac{4\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1, \beta 1} \hat{R}_{x^\beta f}} \quad (\text{C.120})$$

(b) with Q_A^2 and ζ_1 :

$$Q_{A\text{new}}^2 = Q_A^2 \mathcal{R} \frac{4\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1, \gamma 3} \hat{R}_{x^\gamma f}} \quad (\text{C.121})$$

8. A gluon splitting

(a) with Q_\perp^2 and ζ_1 :

$$Q_{\perp\text{new}}^2 = Q_\perp^2 \mathcal{R} \frac{2\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1, \beta 2} \hat{R}_{x^\beta f}} \quad (\text{C.122})$$

(b) with Q_A^2 and ζ_1 :

$$Q_{A\text{new}}^2 = Q_A^2 \mathcal{R} \frac{2\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_1, \gamma 2} \hat{R}_{x^\gamma f}} \quad (\text{C.123})$$

9. K gluon splitting

(a) with Q_\perp^2 and ζ_2 :

$$Q_{\perp\text{new}}^2 = Q_\perp^2 \mathcal{R} \frac{8\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_3, \text{lin}} \hat{R}_{xf}} \quad (\text{C.124})$$

(b) with Q_K^2 and ζ_2 :

$$Q_{K\text{new}}^2 = Q_K^2 \mathcal{R} \frac{8\pi}{\hat{\alpha}_s \mathcal{C}} \frac{1}{I_{\zeta_3, \text{lin}} \hat{R}_{xf}} \quad (\text{C.125})$$

Running of the Coupling: See initial-initial.

C.11 Inverse Transforms

The inversions are for Q_\perp^2 and ζ_1

$$s_{jk} = s_{AK}(\zeta_1 - 1) \quad (\text{C.126})$$

$$s_{aj} = \frac{Q_\perp^2 \zeta_1}{\zeta_1 - 1} \quad (\text{C.127})$$

and for Q_A^2 and ζ_1

$$s_{aj} = \frac{Q_A^2}{N_{sI}} \quad (\text{C.128})$$

$$s_{jk} = (\zeta_1 - 1)s_{AK} \quad (\text{C.129})$$

and for Q_\perp^2 and ζ_2

$$s_{jk} = \frac{Q_\perp^2}{\zeta_2} \quad (\text{C.130})$$

$$s_{aj} = s_{AK}\zeta_2 + Q_\perp^2 \quad (\text{C.131})$$

and for Q_K^2 and ζ_2

$$s_{jk} = Q_K^2 \quad (\text{C.132})$$

$$s_{aj} = \zeta_2 (s_{AK} + Q_K^2) \quad (\text{C.133})$$

and for Q_\perp^2 and ζ_3

$$s_{jk} = \frac{Q_\perp^2}{2\zeta_3} \left(1 + \sqrt{1 + 4 \frac{\zeta_3}{Q_\perp^2} s_{AK}} \right) \quad (\text{C.134})$$

$$s_{aj} = \frac{Q_\perp^2}{2} \left(1 + \sqrt{1 + 4 \frac{\zeta_3}{Q_\perp^2} s_{AK}} \right) . \quad (\text{C.135})$$

D Accept Probabilities

D.1 Helicity Selection

For non maximally helicity violating (MHV) processes see [4]. MHV helicity selection can be simplified by studying the structure of MHV amplitudes, which are discussed in Section D.7 below. MHV amplitudes all have the following form:

$$|M_n^{FC}|_h^2 = |A_0(1^h, \dots, n^h)|^2 \left| \sum_\sigma \frac{1}{\langle \sigma(1)\sigma(2) \rangle \dots \langle \sigma(n)\sigma(1) \rangle} \text{CF}(\sigma(1) \dots \sigma(n)) \right|^2, \quad (\text{D.1})$$

where $|M_n^{FC}|_h^2$ refers to the full colour (FC) MHV squared amplitude with helicity configuration label h , $A_0(1^h, \dots, n^h)$ is some function of the helicities, σ is the relevant set of permutations, and CF is the colour factor. To polarise the hard process we want to calculate if:

$$\frac{\sum_{i=1}^h |M_n^{FC}|_i^2}{\sum_{h'} |M_n^{FC}|_{h'}^2} \geq R, \quad (\text{D.2})$$

for h the helicity-configuration we are currently checking, and the sum over h' is a sum over all helicity configurations, which can be expanded as:

$$\sum_{h'} |M_n^{FC}|_{h'}^2 = \left(\sum_{h'} |A_0(1^{h'}, \dots, n^{h'})|^2 \right) \left| \sum_{\sigma} \frac{1}{\langle \sigma(1)\sigma(2) \rangle \dots \langle \sigma(n)\sigma(1) \rangle} \text{CF}(\sigma(1) \dots \sigma(n)) \right|^2. \quad (\text{D.3})$$

Labelling the second term as $F(\sigma)$, we notice that equation (D.2) now reads:

$$\frac{\sum_{i=1}^h |M_n^{FC}|_i^2}{\sum_{h'} |M_n^{FC}|_{h'}^2} = \frac{\sum_{i=1}^h |A_0(1^i, \dots, n^i)|^2 F(\sigma)}{\sum_{h'} |A_0(1^{h'}, \dots, n^{h'})|^2 F(\sigma)} = \frac{\sum_{i=1}^h |A_0(1^i, \dots, n^i)|^2}{\sum_{h'} |A_0(1^{h'}, \dots, n^{h'})|^2} \geq R, \quad (\text{D.4})$$

and we can therefore use the much simpler expressions $|A_0(1^h, \dots, n^h)|^2 = |M_n^{LC}|_h^2$ to polarise the process. That is, since the colour information is the same for each MHV helicity configuration we can factorise it out from the sum of matrix elements. QCD processes are non-chiral, so we explicitly calculate only half of the factors $|M_n^{LC}|_h^2$, since the other half are equal by parity.

We cannot do the above for 4-quark MHV amplitudes, because there is a second colour-connection when the two quarks have the same helicity. Hence the colour-factor depends on the helicity and cannot be factorised.

D.2 Smooth-Ordering Factor: P_{imp}

Note: this section is largely adapted from the discussion in [5].

In smooth ordering, the only quantity which must still be strictly ordered are the antenna invariant masses, a condition which follows from the nested antenna phase spaces and momentum conservation. Within each individual antenna, and between competing ones, the measure of evolution time is still provided by the ordering variable, which we therefore typically refer to as the “evolution variable” in this context (rather than the “ordering variable”), in order to prevent confusion with the strong-ordering case. The evolution variable can in principle still be chosen to be any of the possibilities given above, though we shall typically use $2p_{\perp}$ for gluon emission and $m_{q\bar{q}}$ for gluon splitting.

In terms of an arbitrary evolution variable, Q , the smooth-ordering factor is [4]

$$P_{\text{imp}}(Q^2, \hat{Q}^2) = \frac{\hat{Q}^2}{\hat{Q}^2 + Q^2}, \quad (\text{D.5})$$

where Q is the evolution scale associated with the current branching, and \hat{Q} measures the scale of the parton configuration before branching. A comparison to the strong-ordering step function

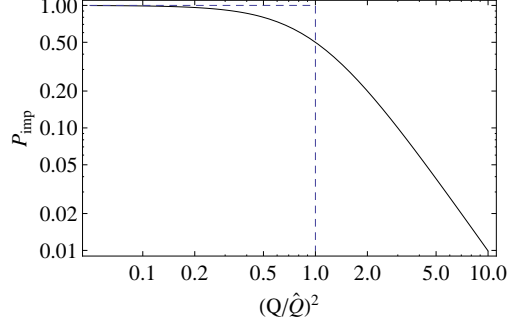


Figure 1: The smooth-ordering factor (*solid*) compared to a strong-ordering Θ function (*dashed*).

is given in fig. 1, on a log-log scale. Since this factor is bounded by $0 \leq P_{\text{imp}} \leq 1$, it can be applied as a simple accept/reject on each trial branching.

When switched on, smooth ordering is technically achieved as follows. After each accepted branching, the daughter antennae involved in that particular branching are allowed to restart their evolution from a scale nominally equivalent to their respective kinematic maximum. Trial branchings are then generated in the “unordered” part of phase space first, for those antennae only, while all other antennae in the event are “on hold”, waiting for the scale to drop back down to normal ordering before the global event evolution is continued. The P_{imp} factor is applied as an extra multiplicative modification to the accept probability for each trial branching, in both the ordered and unordered regions of phase space.

In the strongly-ordered region of phase-space, $Q \ll \hat{Q}$, we may rewrite the P_{imp} factor as

$$P_{\text{imp}} = \frac{1}{1 + \frac{Q^2}{\hat{Q}^2}} \stackrel{Q \ll \hat{Q}}{\approx} 1 - \frac{Q^2}{\hat{Q}^2} + \dots \quad (\text{D.6})$$

Applying this to the $2 \rightarrow 3$ antenna function whose leading singularity is proportional to $1/Q^2$, we see that the overall correction arising from the Q^2/\hat{Q}^2 and higher terms is finite and of order $1/\hat{Q}^2$; a power correction. The LL singular behaviour is therefore not affected. However, at the multiple-emission level, the $1/\hat{Q}^2$ terms do modify the *subleading* logarithmic structure, starting from $\mathcal{O}(\alpha_s^2)$, as we shall return to below.

In the *unordered* region of phase-space, $Q > \hat{Q}$, we rewrite the P_{imp} factor as

$$P_{\text{imp}} = \frac{\hat{Q}^2}{Q^2} \frac{1}{1 + \frac{\hat{Q}^2}{Q^2}} \stackrel{Q > \hat{Q}}{\approx} \frac{\hat{Q}^2}{Q^2} \left(1 - \frac{\hat{Q}^2}{Q^2} + \dots \right), \quad (\text{D.7})$$

which thus effectively modifies the leading singularity of the LL $2 \rightarrow 3$ function from $1/Q^2$ to $1/Q^4$, removing it from the LL counting. The only effective terms $\propto 1/Q^2$ arise from finite terms in the radiation functions, if any such are present, multiplied by the P_{imp} factor. Only a matching to the full tree-level $2 \rightarrow 4$ functions would enable a precise control over these terms. Up to any given fixed order, this can effectively be achieved by matching to tree-level matrix

elements. Matching beyond the fixed-order level is beyond the scope of the current treatment. We thus restrict ourselves to the observation that, at the LL level, smooth ordering is equivalent to strong ordering, with differences only appearing at the subleading level.

The effective $2 \rightarrow 4$ probability in the shower arises from a sum over two different $2 \rightarrow 3 \otimes 2 \rightarrow 3$ histories, each of which has the tree-level form

$$\hat{A} P_{\text{imp}} A \propto \frac{1}{\hat{Q}^2} \frac{\hat{Q}^2}{\hat{Q}^2 + Q^2} \frac{1}{Q^2} = \frac{1}{\hat{Q}^2 + Q^2} \frac{1}{Q^2}, \quad (\text{D.8})$$

thus we may also perceive the combined effect of the modification as the addition of a mass term in the denominator of the propagator factor of the previous splitting. In the strongly ordered region, this correction is negligible, whereas in the unordered region, there is a large suppression from the necessity of the propagator in the previous topology having to be very off-shell, which is reflected by the P_{imp} factor. Using the expansion for the unordered region, eq. (D.7), we also see that the effective $2 \rightarrow 4$ radiation function, obtained from iterated $2 \rightarrow 3$ splittings, is modified as follows,

$$P_{2 \rightarrow 4} \propto \frac{1}{\hat{Q}^2} \frac{\hat{Q}^2}{Q^2} \frac{1}{Q^2} \rightarrow \frac{1}{Q^4} + \mathcal{O}(\dots), \quad (\text{D.9})$$

in the unordered region. That is, the intermediate low scale \hat{Q} , is *removed* from the effective $2 \rightarrow 4$ function, by the application of the P_{imp} factor.

D.3 All-Orders P_{imp} Factor

The path through phase space taken by an unordered shower history is illustrated in fig. 2, from [5]. An antenna starts showering at a scale equal to its invariant mass, \sqrt{s} , and a first $2 \rightarrow 3$ branching occurs at the evolution scale \hat{Q} . This produces the overall Sudakov factor $\Delta_{2 \rightarrow 3}(\sqrt{s}, \hat{Q})$. A daughter antenna, produced by the branching, then starts showering at a scale equal to its own invariant mass, labeled $\sqrt{s_1}$. However, for all scales larger than \hat{Q} , the P_{imp} factor suppresses the evolution in this new dipole so that no leading logs are generated. To leading approximation, the effective Sudakov factor for the combined $2 \rightarrow 4$ splitting is therefore given by

$$\Delta_{2 \rightarrow 4}^{\text{eff}} \sim \Delta_{2 \rightarrow 3}(\sqrt{s}, \hat{Q}), \quad (\text{D.10})$$

in the unordered region. Thus, we see that a dependence on the intermediate scale \hat{Q} still remains in the effective Sudakov factor generated by the smooth-ordering procedure. Since $\hat{Q} < Q$ in the unordered region, the effective Sudakov suppression of these points might be “too strong”. The smooth ordering therefore allows for phase space occupation in regions corresponding to dead zones in a strongly ordered shower, but it does suggest that a correction to the Sudakov factor may be desirable, in the unordered region.

A study of $Z \rightarrow 4$ jets at one loop would be required to shed further light on this question. In the meantime, for all unordered branchings that follow upon a gluon emission, we allow to include a correction to the P_{imp} factor that removes the leading (eikonal) part of the “extra”

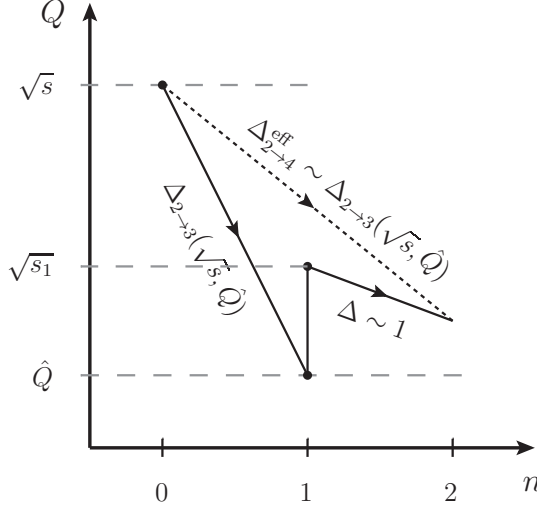


Figure 2: Illustration of scales and Sudakov factors involved in an unordered sequence of two $2 \rightarrow 3$ branchings, representing the smoothly ordered shower's approximation to a hard $2 \rightarrow 4$ process.

Sudakov suppression. We define an all-orders corrected P_{imp} factor as follows:

$$P_{\text{imp}}^{\text{emit}}(Q^2, \hat{Q}^2) \rightarrow \frac{\alpha_s(Q^2)}{\alpha_s(\hat{Q}^2)} \frac{P_{\text{imp}}(Q^2, \hat{Q}^2)}{\Delta_{2 \rightarrow 3}^{\text{eik}}(Q^2, \hat{Q}^2)}, \quad (\text{D.11})$$

with the Eikonal terms of the Sudakov integral given by [5]:

$$\frac{1}{\Delta_{2 \rightarrow 3}^{\text{eik}}(Q^2, \hat{Q}^2)} = \exp \left(\frac{\alpha_s(Q^2)}{2\pi} \mathcal{C} [I_1(\hat{y}) - 2I_2(\hat{y}) - I_1(y) + 2I_2(y)] \right),$$

where \mathcal{C} is the colour factor of the first $2 \rightarrow 3$ branching (the one that produced the intermediate scale \hat{Q}) inside which the unordered $2 \rightarrow 4$ branching is occurring, and $y = Q^2/m_2^2$ ($\hat{y} = \hat{Q}^2/m_2^2$) is the branching scale normalized to the invariant mass squared of that antenna. For evolution in p_\perp (the default for gluon emissions), the I_1 and I_2 integrals are [5]:

$$I_1(y) = \frac{\pi^2}{6} + \frac{1}{2} \ln^2 \left[\frac{y^2}{2(1 + \sqrt{1 - y^2}) - y^2} \right] - \ln^2 \left[\frac{1}{2} (1 + \sqrt{1 - y^2}) \right] - 2 \text{Li}_2 \left[\frac{1}{2} (1 + \sqrt{1 - y^2}) \right] \quad (\text{D.12})$$

$$I_2(y) = - \left(\ln \left[\frac{y^2}{2(1 + \sqrt{1 - y^2}) - y^2} \right] + 2\sqrt{1 - y^2} \right), \quad (\text{D.13})$$

with expressions for other ordering types available in [5]. We note that this factor neglects (positive) collinear-singular terms and (positive) corrections from the running of α_s between Q and \hat{Q} , hence we expect that even this correction factor still only represents a partial compensation, at a level equivalent to removing spurious terms of total order $\alpha_s^3 \ln^3(\hat{Q}^2/Q^2)$ and $\alpha_s^3 \ln^2(\hat{Q}^2/Q^2)$. We also note that a similar factor could be applied

D.4 Gluon Splitting: The Ariadne Factor

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D.5 Matrix-Element Corrections: Leading Colour

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D.6 Matrix-Element Corrections: Full Colour

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D.7 Matrix-Element Corrections: MHV amplitudes

For fast evaluation of certain types of helicity configuration VINCIA uses Maximally Helicity Violating (MHV) amplitudes. MHV amplitudes have the advantage of being an analytical solution for n partons which is independent of Feynman diagrams. In the following we consider all particles to be outgoing and massless.

To be in the MHV configuration all but two particles must have the same helicity. We define our spinors as:

$$u_{\pm}(p) = \frac{1}{2} (1 \pm \gamma^5) u(p) , \quad \overline{u_{\pm}(p)} = \overline{u(p)} \frac{1}{2} (1 \mp \gamma^5) , \quad (\text{D.14})$$

together with their inner products:

$$\overline{u_{-}(i)} u_{+}(j) \equiv \langle ij \rangle = \sqrt{p_j^{+}} e^{i\phi_i} - \sqrt{p_i^{+}} e^{i\phi_j} , \quad (\text{D.15})$$

$$\overline{u_{+}(i)} u_{-}(j) \equiv [ij] = \langle ji \rangle^{*} , \quad (\text{D.16})$$

where $p_i^{+} = p_i^0 + p_i^3$ and $e^{i\phi_i} = \frac{p_i^1 + ip_i^2}{\sqrt{p_i^{+}}}$. For more details about spinor inner products and their properties see [6]. Note that in recent literature one often finds the convention $[ij] = \langle ij \rangle^{*}$, which is different to above. Any future formulae/spinor-helicity properties borrowed from literature should bare this in mind.

The MHV amplitudes used in VINCIA are all colour-ordered. We use a different QCD convention for MHV than in the rest of VINCIA. For MHV amplitudes the QCD Casimirs become $T_R = 1$, $C_F = 8/3$, and $C_A = 3$, which affects the colour-algebra as seen in [6]. The n -gluon

MHV amplitude with negative-helicity gluons at positions i and j is given by the Parke Taylor formula [7]:

$$A_n(i^-, j^-) = i \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \dots \langle n1 \rangle} . \quad (\text{D.17})$$

There is an equally compact MHV formula for the process involving $n - 2$ gluons and a quark anti-quark pair. If the quark and gluon i each have negative helicity, and the anti-quark and all other gluons have positive helicity, then the amplitude is [8]:

$$A_n(q^-, i^-, \bar{q}^+) = \frac{\langle qi \rangle^3 \langle \bar{q}i \rangle}{\langle \bar{q}q \rangle \langle q1 \rangle \langle 12 \rangle \dots \langle (n-2)\bar{q} \rangle} , \quad (\text{D.18})$$

where the numbers refer to the (colour ordered) gluons. If we exchange the helicities on the quarks, it is sufficient to exchange the exponents in the numerator. See [8] for the four-quark, n-gluon MHV amplitude; as well as the two-quark, two-lepton, n-gluon MHV amplitude. For each amplitude, exchanging the helicity of each particle corresponds to exchanging $\langle ij \rangle \rightarrow [ji]$.

The MHV amplitudes involving the exchange of a W boson still need testing, but have been written into MHV.cc

When doing the helicity-clustering, an MHV configuration will always cluster back into either an unphysical helicity state, or into an MHV state. This allows for quick matrix-element corrections of complex states such as the 7-gluon state. The MHV configurations also provide the dominant contributions to a helicity-summed amplitude. It may therefore be useful to give the user the option to include MHV corrections for very high-multiplicity states.

D.8 Matrix-Element Corrections: Different Interfering Borns

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E Cutoff Boundaries

E.1 Fixed Transverse Momentum

- Consider the region defined by $y_{ij}y_{jk} \geq y_\perp$. For illustration, a value of $y_\perp = 0.1$ was used for the contour shown with light green shading in fig. 3.
- The (larger) invariant-mass region that completely encloses the y_\perp one is defined by $y_D = \min(y_{ij}, y_{jk}) \geq \frac{1}{2}(1 - \sqrt{1 - 4y_\perp})$. This is shown with light blue shading in fig. 3.
- The (smaller) invariant-mass region that is completely enclosed by the y_\perp one is defined by $y_D = \min(y_{ij}, y_{jk}) \geq \sqrt{y_\perp}$. This is shown with light yellow shading in fig. 3.

To translate this to evolution variables, with arbitrary normalization factors, use $y_\perp = Q_\perp^2/s_{IK}/N_\perp$ and $m_D^2/s_{IK}/N_D$.

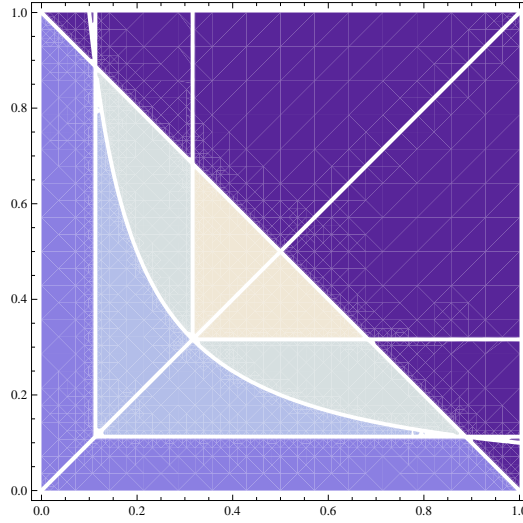


Figure 3: Illustration of the intersection/nesting of p_T and m_D contours.

E.2 Fixed Dipole Mass

- Consider the region defined by $\min(y_{ij}y_{jk}) \geq y_D$, with y_D some fixed value.
- The (larger) transverse-momentum region that completely encloses the y_D one is defined by $y_{\perp} = y_{ij}y_{jk} \geq y_D^2$. This relationship is illustrated by the light-green and light-yellow shaded regions in fig. 3.
- The (smaller) transverse-momentum region that is completely enclosed by the y_D one is defined by $y_{\perp} = y_{ij}y_{jk} \geq \frac{1}{4}(1 - (1 - 2y_D)^2)$. This relationship is illustrated by the light-green and light-blue shaded regions in fig. 3.

To translate this to evolution variables, with arbitrary normalization factors, use $y_{\perp} = Q_{\perp}^2/s_{IK}/N_{\perp}$ and $m_D^2/s_{IK}/N_D$.

References

- [1] Walter T. Giele, David A. Kosower, and Peter Z. Skands. A Simple shower and matching algorithm. *Phys.Rev.*, D78:014026, 2008.
- [2] W.T. Giele, D.A. Kosower, and P.Z. Skands. Higher-Order Corrections to Timelike Jets. *Phys.Rev.*, D84:054003, 2011.
- [3] Aude Gehrmann-De Ridder, Mathias Ritzmann, and Peter Skands. Timelike Dipole-Antenna Showers with Massive Fermions. *Phys.Rev.*, D85:014013, 2012.
- [4] Andrew J. Larkoski, Juan J. Lopez-Villarejo, and Peter Skands. Helicity-Dependent Showers and Matching with VINCIA. *Phys.Rev.*, D87(5):054033, 2013.

- [5] L. Hartgring, E. Laenen, and P. Skands. Antenna Showers with One-Loop Matrix Elements. *JHEP*, 1310:127, 2013.
- [6] Lance J. Dixon. Calculating scattering amplitudes efficiently. arXiv:hep-ph/9601359v2, 1996.
- [7] Stephen J. Parke and T.R. Taylor. An Amplitude for n Gluon Scattering. *Phys. Rev. Lett.*, 56:2459, 1986.
- [8] Michelangelo L. Mangano and Stephen J. Parke. Quark - Gluon Amplitudes in the Dual Expansion. *Nucl. Phys.*, B299:673, 1988.