

VINCIA Authors' Compendium

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A Evolution Equations

A.1 Final–Final Evolution Variables

The evolution variables considered in VINCIA for final–final antennae are the following [1, 2]:

$$Q_{\perp}^2 = N_{\perp} \frac{s_{ij}s_{jk}}{m_{IK}^2} = N_{\perp} y_{ij} y_{jk} m_{IK}^2 = N_{\perp} p_{\perp A}^2, \quad (1)$$

$$m_D^2 = N_D \min(s_{ij}, s_{jk}) = N_D \min(y_{ij}, y_{jk}) m_{IK}^2, \quad (2)$$

$$E^{*2} = \frac{(s_{ij} + s_{jk})^2}{m_{IK}^2} = (y_{ij} + y_{jk})^2 m_{IK}^2, \quad (3)$$

$$m_{g^*}^2 = m_{jk}^2, \quad (4)$$

with the arbitrary normalization factors $N_{\perp} \in [1, 4]$ and $N_D \in [1, 2]$, the invariant mass

$$m_{IK}^2 = (p_I + p_K)^2 = (p_i + p_j + p_k)^2, \quad (5)$$

and the symbol s_{ij} defined as the dot product

$$s_{ij} \equiv 2p_i \cdot p_j = (p_i + p_j)^2 - m_i^2 - m_j^2 \stackrel{m=0}{=} m_{ij}^2. \quad (6)$$

The maximum values that these evolution variables attain on the physical final-final antenna phase-space are:

$$Q_{\perp \max}^2 = \frac{N_{\perp}}{4} m_{IK}^2, \quad (7)$$

$$m_{D \max}^2 = \frac{N_D}{2} m_{IK}^2, \quad (8)$$

$$E_{\max}^{*2} = m_{IK}^2, \quad (9)$$

$$m_{g^* \max}^2 = m_{IK}^2. \quad (10)$$

Note on dimensionality: the dimensionless form of the evolution variable is $y = Q^2/m_{IK}^2$, with Q denoting the choice of evolution variable among the above possibilities. Throughout, we use the notation y for scaled dot products, $y_{ij} = s_{ij}/m_{IK}^2$, and y' for scaled invariant masses, $y'_{ij} = m_{ij}^2/m_{IK}^2$.

Note for m_D : the expressions below correspond to the branch with $y_{ij} < y_{jk}$ and hence will only generate branchings over half of phase space. For trial antenna functions symmetric in the invariants (specifically the soft-eikonal and hard-finite ones, see sec. A.4), the trial generation is

done by multiplying the kernel by a factor 2 and randomly keeping or swapping the generated invariants. For the I - and K -collinear sector terms, we use that they are mutually related by $i \leftrightarrow k$, and hence an I -collinear term over all of phase space can be composed from an I -collinear one on the branch $y_{ij} < y_{jk}$ combined with a K -collinear one with swapped invariants on the complementary branch.

A.2 Zeta Definitions

The following choices of ζ are used:

$$\zeta_1 = \frac{y_{ij}}{y_{ij} + y_{jk}} \quad (11)$$

$$\zeta_2 = y_{ij} \quad (12)$$

$$\zeta_3 = y_{jk} . \quad (13)$$

The final-final phase-space limits are, for Q_\perp :

$$\zeta_{1\pm}(Q_\perp^2) = \frac{1}{2} \left(1 \pm \sqrt{1 - \frac{4}{N_\perp} \frac{Q_\perp^2}{m_{IK}^2}} \right) = \frac{1}{2} (1 \pm \sqrt{1 - 4 y_{ij} y_{jk}}) , \quad (14)$$

$$\zeta_{2\pm}(Q_\perp^2) = \zeta_{1\pm}(Q_\perp^2) , \quad (15)$$

$$\zeta_{3\pm}(Q_\perp^2) = \zeta_{1\pm}(Q_\perp^2) , \quad (16)$$

for m_D :

$$\zeta_{1-}(m_D^2) = \frac{1}{2} , \quad (17)$$

$$\zeta_{1+}(m_D^2) = 1 - \frac{m_D^2}{N_D m_{IK}^2} = 1 - y_{ij} , \quad (18)$$

$$\zeta_{2-}(m_D^2) = \text{N/A} , \quad (19)$$

$$\zeta_{2+}(m_D^2) = \text{N/A} , \quad (20)$$

$$\zeta_{3-}(m_D^2) = \frac{m_D^2}{N_D m_{IK}^2} = y_{ij} , \quad (21)$$

$$\zeta_{3+}(m_D^2) = 1 - \frac{m_D^2}{N_D m_{IK}^2} = 1 - y_{ij} , \quad (22)$$

for E^* :

$$\zeta_{\pm}(E^{*2}) = \text{Special: see below} \quad (23)$$

for m_{g^*} :

$$\zeta_{2-}(m_{q\bar{q}}^2) = 0 , \quad (24)$$

$$\zeta_{2+}(m_{q\bar{q}}^2) = 1 - \frac{m_{g^*}^2}{m_{IK}^2} = 1 - y'_{jk} , \quad (25)$$

using the definition $y'_{jk} = m_{jk}^2/m_{IK}^2$ in the last expression. Note that the phase-space limits for E^* coincide with the collinear limits. Integrations over any finite interval of E^* over the full allowed ζ range would therefore yield infinities. When using E^* -ordering, it is necessary to impose a hadronization cutoff in a complementary variable, such as Q_{\perp} or m_D . This cutoff then defines the ζ boundaries for the integrations.

A.3 Jacobians

The Jacobians for the transformation from the original LIPS variables, (s_{ij}, s_{jk}) , to the shower variables, (Q^2, ζ) , are written as a product of a normalization-and- Q -dependent piece and a ζ -dependent factor,

$$|J| = J_Q \times J_{\zeta} . \quad (26)$$

They are, for Q_\perp^2 :

$$|J(Q_\perp^2, \zeta_1)| = \frac{1}{2N_\perp} \times \frac{m_{IK}^2}{\zeta_1(1 - \zeta_1)} , \quad (27)$$

$$|J(Q_\perp^2, \zeta_2)| = \frac{1}{N_\perp} \times \frac{m_{IK}^2}{\zeta_2} , \quad (28)$$

$$|J(Q_\perp^2, \zeta_3)| = \frac{1}{N_\perp} \times \frac{m_{IK}^2}{\zeta_3} , \quad (29)$$

for m_D^2 :

$$|J(m_D^2, \zeta_1)| = \text{Not Used} , \quad (30)$$

$$|J(m_D^2, \zeta_2)| = \text{N/A} , \quad (31)$$

$$|J(m_D^2, \zeta_3)| = \frac{1}{N_D} \times m_{IK}^2 , \quad (32)$$

for E^{*2} :

$$|J(E^{*2}, \zeta_1)| = \frac{1}{2} \times m_{IK}^2 , \quad (33)$$

$$|J(E^{*2}, \zeta_2)| = \frac{m_{IK}}{2\sqrt{E^{*2}}} \times m_{IK}^2 , \quad (34)$$

$$|J(E^{*2}, \zeta_3)| = \frac{m_{IK}}{2\sqrt{E^{*2}}} \times m_{IK}^2 , \quad (35)$$

for m_{g^*} :

$$|J(m_{q\bar{q}}^2, \zeta_2)| = 1 \times m_{IK}^2 . \quad (36)$$

A.4 Trial Functions

The following trial functions are available:

$$\text{Eikonal (soft)} : \hat{a}_E = \frac{1}{m_{IK}^2} \frac{2}{y_{ij} y_{jk}} \quad (37)$$

$$\text{Constant (hard)} : \hat{a}_F = \frac{1}{m_{IK}^2} \quad (38)$$

$$\text{I Collinear (sector)} : \hat{a}_I = \frac{1}{m_{IK}^2} \frac{2}{y_{ij}(1 - y_{jk})} \quad (39)$$

$$\text{K Collinear (sector)} : \hat{a}_K = \frac{1}{m_{IK}^2} \frac{2}{y_{jk}(1 - y_{ij})} \quad (40)$$

$$\text{K Splitting } (g \rightarrow q\bar{q}) : \hat{a}_S = \frac{1}{m_{q\bar{q}}^2} = \frac{1}{m_{IK}^2} \frac{1}{y'_{jk}}, \quad (41)$$

where we emphasize that y'_{jk} in the last expression is defined by $y'_{jk} = m_{jk}^2/m_{IK}^2$.

A.5 Zeta Integrals

For a given trial antenna function, \hat{a} , the definition of the ζ integral is:

$$I_\zeta = \int_{\zeta_a}^{\zeta_b} d\zeta J_\zeta \hat{a} \quad (42)$$

where $|J_\zeta|$ signifies the part of the Jacobian that only has ζ dependence (see above), and $\zeta_b > \zeta_a$ represents an arbitrary ζ interval. This interval will in general be larger than the physically allowed one (trials generated outside the physical phase space will be rejected by a veto). We shall nevertheless still assume that all ζ values are at least inside the range $\zeta \in [0, 1]$.

The integration kernels are, for Q_\perp :

$$J_\zeta \hat{a}_{E,F}(Q_\perp^2, \zeta_1) = \frac{1}{(1 - \zeta_1)\zeta_1}, \quad (43)$$

$$J_\zeta \hat{a}_I(Q_\perp^2, \zeta_3) = \frac{1}{(1 - \zeta_3)}, \quad (44)$$

$$J_\zeta \hat{a}_K(Q_\perp^2, \zeta_2) = \frac{1}{(1 - \zeta_2)}, \quad (45)$$

for m_D :

$$J_\zeta \hat{a}_E(m_D^2, \zeta_3) = \frac{1}{\zeta_3} , \quad (46)$$

$$J_\zeta \hat{a}_F(m_D^2, \zeta_3) = 1 , \quad (47)$$

$$J_\zeta \hat{a}_I(m_D^2, \zeta_3) = \frac{1}{(1 - \zeta_3)} , \quad (48)$$

for E^* :

$$J_\zeta \hat{a}_E(E^{*2}, \zeta_1) = \frac{1}{\zeta_1^2} , \quad (49)$$

$$J_\zeta \hat{a}_F(E^{*2}, \zeta_1) = \frac{1}{2} , \quad (50)$$

$$(51)$$

for m_{g^*} :

$$J_\zeta \hat{a}_S(m_{g^*}^2, \zeta_2) = 1 \quad (52)$$

with integrals over the range $\zeta_a < \zeta_b$, for Q_\perp :

$$I_{\zeta E, F}(Q_\perp^2, \zeta_1) = \ln \left(\frac{\zeta_b(1 - \zeta_a)}{\zeta_a(1 - \zeta_b)} \right) , \quad (53)$$

$$I_{\zeta I}(Q_\perp^2, \zeta_3) = \ln \left(\frac{1 - \zeta_a}{1 - \zeta_b} \right) , \quad (54)$$

$$I_{\zeta K}(Q_\perp^2, \zeta_2) = \ln \left(\frac{1 - \zeta_a}{1 - \zeta_b} \right) , \quad (55)$$

for m_D :

$$I_{\zeta E}(m_D^2, \zeta_3) = \ln \left(\frac{\zeta_b}{\zeta_a} \right) , \quad (56)$$

$$I_{\zeta F}(m_D^2, \zeta_3) = \zeta_b - \zeta_a , \quad (57)$$

$$I_{\zeta I}(m_D^2, \zeta_3) = \ln \left(\frac{1 - \zeta_a}{1 - \zeta_b} \right) , \quad (58)$$

for E^* :

$$I_{\zeta E}(E^{*2}, \zeta_1) = \frac{1}{z_a} - \frac{1}{z_b} , \quad (59)$$

$$I_{\zeta F}(E^{*2}, \zeta_1) = \zeta_b - \zeta_a , \quad (60)$$

for m_{g^*} :

$$I_{\zeta S}(m_{g^*}^2, \zeta_2) = \zeta_b - \zeta_a . \quad (61)$$

A.6 Evolution Integrals

The evolution integral, for a particular choice of Q and ζ , is defined as follows

$$\hat{\mathcal{A}}(Q_1^2, Q_2^2) = \int_{Q_2^2}^{Q_1^2} dQ^2 \frac{\mathcal{C} g_s^2}{16\pi^2 m_{IK}^2} J_Q(Q, \zeta) I_\zeta(Q, \zeta) , \quad (62)$$

with \mathcal{C} the (trial) color factor (typically C_A for gluon emission and 1 for gluon splitting) and J_Q the non- ζ dependent part of the Jacobian, see eqs. (27) – (36).

Note: for massive partons, the phase-space factor should actually be larger: m_{IK} in the denominator should be replaced by the Källén function [3]:

$$m_{IK}^2 \rightarrow \lambda(m_{IK}^2, m_I^2, m_K^2) = m_{IK}^4 + m_I^4 + m_K^4 - 2(m_{IK}^2 m_I^2 + m_{IK}^2 m_K^2 + m_I^2 m_K^2) . \quad (63)$$

For massive partons, this is taken care of during trial generation by applying an overall prefactor representing the phase-space volume and using the same integrals as shown here below.

We label the integrand in the above equation by

$$d\hat{\mathcal{A}} = \frac{\mathcal{C} g_s^2}{16\pi^2 s} J_Q(Q, \zeta) I_\zeta(Q, \zeta) , \quad (64)$$

which takes the following specific forms, for Q_\perp :

$$d\hat{\mathcal{A}}_E(Q_\perp^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta E}(Q_\perp^2, \zeta_1) \frac{1}{Q_\perp^2} , \quad (65)$$

$$d\hat{\mathcal{A}}_F(Q_\perp^2) = \frac{1}{2N_\perp} \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta F}(Q_\perp^2, \zeta_1) \frac{1}{m_{IK}^2} , \quad (66)$$

$$d\hat{\mathcal{A}}_I(Q_\perp^2) = 2 \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta I}(Q_\perp^2, \zeta_3) \frac{1}{Q_\perp^2} , \quad (67)$$

$$d\hat{\mathcal{A}}_K(Q_\perp^2) = 2 \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta K}(Q_\perp^2, \zeta_2) \frac{1}{Q_\perp^2} , \quad (68)$$

for m_D :

$$d\hat{\mathcal{A}}_{E,I}(m_D^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} 2I_{\zeta E,I}(m_D^2, \zeta_3) \frac{1}{m_D^2} , \quad (69)$$

$$d\hat{\mathcal{A}}_F(m_D^2) = \frac{1}{N_D} \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta F}(m_D^2, \zeta_3) \frac{1}{m_{IK}^2} , \quad (70)$$

for E^* :

$$d\hat{\mathcal{A}}_E(E^{*2}) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta E}(E^{*2}, \zeta_1) \frac{1}{\sqrt{E^{*2} m_{IK}^2}} , \quad (71)$$

$$d\hat{\mathcal{A}}_F(E^{*2}) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} \frac{1}{2} I_{\zeta F}(E^{*2}, \zeta_1) \frac{1}{m_{IK}^2} , \quad (72)$$

for m_{g^*} :

$$d\hat{\mathcal{A}}_S(m_{g^*}^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta S}(m_{g^*}^2, \zeta_2) \frac{1}{m_{g^*}^2} . \quad (73)$$

For a constant trial $\hat{\alpha}_s$, the evolution integrals are, for Q_\perp :

$$\hat{\mathcal{A}}_E^0(Q_{\perp 1}^2, Q_{\perp 2}^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta E} \ln \left(\frac{Q_{\perp 1}^2}{Q_{\perp 2}^2} \right) , \quad (74)$$

$$\hat{\mathcal{A}}_F^0(Q_{\perp 1}^2, Q_{\perp 2}^2) = \frac{1}{2N_\perp} \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta F} \frac{(Q_{\perp 1}^2 - Q_{\perp 2}^2)}{m_{IK}^2} , \quad (75)$$

$$\hat{\mathcal{A}}_{I,K}^0(Q_{\perp 1}^2, Q_{\perp 2}^2) = 2 \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta I,K} \ln \left(\frac{Q_{\perp 1}^2}{Q_{\perp 2}^2} \right) , \quad (76)$$

for m_D :

$$\hat{\mathcal{A}}_{E,I}^0(m_{D1}^2, m_{D2}^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} 2I_{\zeta E,I} \ln \left(\frac{m_{D1}^2}{m_{D2}^2} \right) , \quad (77)$$

$$\hat{\mathcal{A}}_F^0(m_{D1}^2, m_{D2}^2) = \frac{1}{N_D} \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta F} \frac{(m_{D1}^2 - m_{D2}^2)}{m_{IK}^2} , \quad (78)$$

for E^* :

$$\hat{\mathcal{A}}_E^0(E_1^{*2}, E_2^{*2}) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} 2I_{\zeta E} \frac{(\sqrt{E_1^{*2}} - \sqrt{E_2^{*2}})}{m_{IK}} , \quad (79)$$

$$\hat{\mathcal{A}}_F^0(E_1^{*2}, E_2^{*2}) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} \frac{1}{2} I_{\zeta F} \frac{(E_1^{*2} - E_2^{*2})}{m_{IK}^2} \quad (80)$$

for m_{g^*} :

$$\hat{\mathcal{A}}_S^0(m_{g1}^2, m_{g2}^2) = \frac{\hat{\alpha}_s}{4\pi} \mathcal{C} I_{\zeta S} \ln \left(\frac{m_{g1}^2}{m_{g2}^2} \right) . \quad (81)$$

For a first-order running trial $\hat{\alpha}_s(Q^2)$,

$$\hat{\alpha}_s(Q_\perp^2) = \frac{1}{b_0 \ln \left(\frac{k_R^2 p_{\perp A}^2}{\Lambda^2} \right)} = \frac{1}{b_0 \ln \left(\frac{k_R^2 Q_\perp^2}{N_\perp \Lambda^2} \right)}, \quad (82)$$

$$\hat{\alpha}_s(m_D^2) = \frac{1}{b_0 \ln \left(\frac{k_R^2 m_{\min}^2}{\Lambda^2} \right)} = \frac{1}{b_0 \ln \left(\frac{k_R^2 m_D^2}{N_D \Lambda^2} \right)}, \quad (83)$$

$$\hat{\alpha}_s(m_{g^*}^2) = \frac{1}{b_0 \ln \left(\frac{k_R^2 m_{g^*}^2}{\Lambda^2} \right)}, \quad (84)$$

with k_R an arbitrary scaling factor that includes the compound effect of any renormalization-scale prefactor choices and the optional translation between the MSbar and CMW schemes for Λ , the evolution integrals are, for Q_\perp :

$$\hat{\mathcal{A}}_E^1(Q_{\perp 1}^2, Q_{\perp 2}^2) = \frac{CI_{\zeta E}}{4\pi b_0} \ln \left(\frac{\ln \left(\frac{k_R^2 Q_{\perp 1}^2}{N_\perp \Lambda^2} \right)}{\ln \left(\frac{k_R^2 Q_{\perp 2}^2}{N_\perp \Lambda^2} \right)} \right), \quad (85)$$

$$\hat{\mathcal{A}}_F^1(Q_{\perp 1}^2, Q_{\perp 2}^2) = \text{Not Used (generates LogIntegrals)}, \quad (86)$$

$$\hat{\mathcal{A}}_{I,K}^1(Q_{\perp 1}^2, Q_{\perp 2}^2) = 2 \frac{CI_{\zeta I,K}}{4\pi b_0} \ln \left(\frac{\ln \left(\frac{k_R^2 Q_{\perp 1}^2}{N_\perp \Lambda^2} \right)}{\ln \left(\frac{k_R^2 Q_{\perp 2}^2}{N_\perp \Lambda^2} \right)} \right), \quad (87)$$

for m_D :

$$\hat{\mathcal{A}}_{E,I}^1(m_{D1}^2, m_{D2}^2) = 2 \frac{CI_{\zeta E,I}}{4\pi b_0} \ln \left(\frac{\ln \left(\frac{k_R^2 m_{D1}^2}{N_D \Lambda^2} \right)}{\ln \left(\frac{k_R^2 m_{D2}^2}{N_D \Lambda^2} \right)} \right), \quad (88)$$

$$\hat{\mathcal{A}}_F^1(m_{D1}^2, m_{D2}^2) = \text{Not Used (generates LogIntegrals)}, \quad (89)$$

for m_{g^*} :

$$\hat{\mathcal{A}}_S^1(m_{g1}^2, m_{g2}^2) = \frac{CI_{\zeta S}}{4\pi b_0} \ln \left(\frac{\ln \left(\frac{k_R^2 m_{g1}^2}{\Lambda^2} \right)}{\ln \left(\frac{k_R^2 m_{g2}^2}{\Lambda^2} \right)} \right). \quad (90)$$

A.7 Generation of Trial Evolution Scale

The trial Sudakov factor is defined as:

$$\hat{\Delta}(Q_1^2, Q_2^2) = \exp \left[-\hat{\mathcal{A}}(Q_1^2, Q_2^2) \right], \quad (91)$$

and the next trial scale is found by solving the equation:

$$\mathcal{R} = \hat{\Delta}(Q^2, Q_{\text{new}}^2) , \quad (92)$$

for Q_{new} , with \mathcal{R} a random number distributed uniformly in the interval $\mathcal{R} \in [0, 1]$, and Q the current “restart scale”. For strongly ordered showers, the restart scale after an accepted trial branching is the evolution scale evaluated on the current parton configuration. For smoothly ordered showers, this restart scale is only used for antennae that are not color-adjacent to the branching that occurred; for the newly created antennae, and (optionally) for any color-adjacent ones, the restart scale is the respective antenna invariant masses¹.

For both strongly and smoothly ordered showers, the restart scale after a failed (vetoed) trial branching is the scale of the failed branching.

Note: to optimize event generation, trial scales can be saved and reused for any antennae whose flavors, spins, and invariant masses are preserved by the preceding branching step.

For constant trial $\hat{\alpha}_s$, the solutions for the next trial scale are, for Q_{\perp} :

$$Q_{\perp E \text{new}}^2 = Q_{\perp}^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta E}}} , \quad (93)$$

$$Q_{\perp F \text{new}}^2 = Q_{\perp}^2 - m_{IK}^2 2N_{\perp} \frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta F}} \ln(1/\mathcal{R}) , \quad (94)$$

$$Q_{\perp I, K \text{new}}^2 = Q_{\perp}^2 \mathcal{R}^{\frac{1}{2} \frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta I, K}}} , \quad (95)$$

for m_D :

$$m_{DE, I \text{new}}^2 = m_D^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s} \frac{1}{2CI_{\zeta E, I}}} , \quad (96)$$

$$m_{DF \text{new}}^2 = m_D^2 - m_{IK}^2 N_D \frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta F}} \ln(1/\mathcal{R}) , \quad (97)$$

for E^* :

$$E_{E \text{new}}^{*2} = \left(\sqrt{E^{*2}} - m_{IK} \frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta E}} \ln(1/\mathcal{R}) \right)^2 , \quad (98)$$

$$E_{F \text{new}}^{*2} = E^{*2} - m_{IK}^2 \frac{4\pi}{\hat{\alpha}_s} \frac{1}{2CI_{\zeta F}} \ln(1/\mathcal{R}) , \quad (99)$$

for m_D :

$$m_{g^* S \text{new}}^2 = m_{g^*}^2 \mathcal{R}^{\frac{4\pi}{\hat{\alpha}_s} \frac{1}{CI_{\zeta S}}} . \quad (100)$$

¹This allows hard $2 \rightarrow n$ branchings to be generated inside the newly created antennae (and optionally within the color-adjacent ones) without disturbing the evolution of the rest of the event.

For a one-loop running trial $\hat{\alpha}_s(\mu_R^2)$, with $\mu_R^2 \propto Q^2$, the solutions for the next trial scale are, for Q_\perp :

$$\ln \left(\frac{k_R^2 Q_{\perp E \text{new}}^2}{N_\perp \Lambda^2} \right) = \mathcal{R}^{\frac{4\pi b_0}{\bar{c} I_{\zeta E}}} \ln \left(\frac{k_R^2 Q_\perp^2}{N_\perp \Lambda^2} \right), \quad (101)$$

$$\ln \left(\frac{k_R^2 Q_{\perp I, K \text{new}}^2}{N_\perp \Lambda^2} \right) = \mathcal{R}^{\frac{1}{2} \frac{4\pi b_0}{\bar{c} I_{\zeta E}}} \ln \left(\frac{k_R^2 Q_\perp^2}{N_\perp \Lambda^2} \right), \quad (102)$$

for m_D :

$$\ln \left(\frac{k_R^2 m_{DE, I \text{new}}^2}{N_D \Lambda^2} \right) = \mathcal{R}^{\frac{4\pi b_0}{2\bar{c} I_{\zeta E, I}}} \ln \left(\frac{k_R^2 m_D^2}{N_D \Lambda^2} \right), \quad (103)$$

for m_{g^*} :

$$\ln \left(\frac{k_R^2 m_{g^* S \text{new}}^2}{\Lambda^2} \right) = \mathcal{R}^{\frac{4\pi b_0}{\bar{c} I_{\zeta S}}} \ln \left(\frac{k_R^2 m_{g^*}^2}{\Lambda^2} \right). \quad (104)$$

A.8 Generation of Trial Zeta

The trial value for ζ is found by inverting the equation

$$\mathcal{R}_\zeta = \frac{I_\zeta(\zeta_{\min}, \zeta)}{I_\zeta(\zeta_{\min}, \zeta_{\max})}, \quad (105)$$

where the boundary values $(\zeta_{\min}, \zeta_{\max})$ must be the same as those that were used to evaluate the I_ζ integrals during the generation of the trial scale above, i.e., they must correspond to the phase-space overestimate used for the trial generation. The forms of I_ζ are given for each evolution variable separately in eqs. (53)–(60).

For Q_\perp , the solutions to eq. (105) are:

$$\zeta_{1E, F}(\mathcal{R}) = \left[1 + \frac{1 - \zeta_{\min}}{\zeta_{\min}} \left(\frac{\zeta_{\min}(1 - \zeta_{\max})}{\zeta_{\max}(1 - \zeta_{\min})} \right)^{\mathcal{R}} \right]^{-1}, \quad (106)$$

$$\zeta_{3I}(\mathcal{R}) = \zeta_{2K}(\mathcal{R}) = 1 - (1 - \zeta_{\min}) \left(\frac{1 - \zeta_{\max}}{1 - \zeta_{\min}} \right)^{\mathcal{R}}, \quad (107)$$

for m_D :

$$\zeta_{3E}(\mathcal{R}) = \zeta_{\min} \left(\frac{\zeta_{\max}}{\zeta_{\min}} \right)^{\mathcal{R}}, \quad (108)$$

$$\zeta_{3F}(\mathcal{R}) = \zeta_{\min} + \mathcal{R}(\zeta_{\max} - \zeta_{\min}), \quad (109)$$

$$\zeta_{3I}(\mathcal{R}) = 1 - (1 - \zeta_{\min}) \left(\frac{1 - \zeta_{\max}}{1 - \zeta_{\min}} \right)^{\mathcal{R}}, \quad (110)$$

for E^* :

$$\zeta_{1E}(\mathcal{R}) = \frac{\zeta_{\max}\zeta_{\min}}{\zeta_{\max} - \mathcal{R}(\zeta_{\max} - \zeta_{\min})} , \quad (111)$$

$$\zeta_{1F}(\mathcal{R}) = \zeta_{\min} + \mathcal{R}(\zeta_{\max} - \zeta_{\min}) , \quad (112)$$

for m_{g^*} :

$$\zeta_{2S}(\mathcal{R}) = \zeta_{\min} + \mathcal{R}(\zeta_{\max} - \zeta_{\min}) . \quad (113)$$

A.9 Accept of Trial Zeta: Massless Phase-Space Boundaries

The generated value of ζ can now be compared to the limits imposed by the physical phase space at the generated value of Q and a rejection imposed if the generated ζ value falls outside the phase space, cf. eqs. (14)–(25).

A.10 Inverse Transforms

After a set of shower variables has been generated, the Q^2 and ζ choices must be inverted to reobtain the branching invariants, (s_{ij}, s_{jk}) , which are used to construct the kinematics of the trial branching. These inversions are, for Q_{\perp} :

$$\begin{aligned} & \begin{matrix} Q_{\perp}^2 & \zeta_1 & \zeta_2 & \zeta_3 \end{matrix} \\ y_{ij} &= \sqrt{\frac{\zeta_1}{1-\zeta_1}} \sqrt{\frac{Q_{\perp}^2}{N_{\perp} m_{IK}^2}} & \zeta_2 & \frac{1}{\zeta_3} \frac{Q_{\perp}^2}{N_{\perp} m_{IK}^2} , \\ y_{jk} &= \sqrt{\frac{1-\zeta_1}{\zeta_1}} \sqrt{\frac{Q_{\perp}^2}{N_{\perp} m_{IK}^2}} & \frac{1}{\zeta_2} \frac{Q_{\perp}^2}{N_{\perp} m_{IK}^2} & \zeta_3 , \end{aligned} \quad (114)$$

for m_D (on $y_{ij} < y_{jk}$ branch):

$$\begin{aligned} & \begin{matrix} m_D^2 & \zeta_3 \end{matrix} \\ y_{ij} &= \frac{m_D^2}{N_D} , \\ y_{jk} &= \zeta_3 , \end{aligned} \quad (115)$$

for E^* :

$$\begin{aligned} & \begin{matrix} E^{*2} & \zeta_1 \end{matrix} \\ y_{ij} &= \zeta_1 \sqrt{\frac{E^{*2}}{m_{IK}^2}} , \\ y_{jk} &= (1 - \zeta_1) \sqrt{\frac{E^{*2}}{m_{IK}^2}} , \end{aligned} \quad (116)$$

for $m_{q\bar{q}}^2$:

$$\begin{aligned} m_{q\bar{q}}^2 &= \zeta_2 \\ y_{ij} &= \zeta_2 , \\ y'_{jk} &= \frac{m_{q\bar{q}}^2}{m_{IK}^2} , \end{aligned} \tag{117}$$

A.11 Mass Corrections for Light Quarks (u,d,s,c,b)

By default, VINCIA treats all the lightest 5 quark flavours as massless. However, it is still possible to enable several corrections that approximate mass effects. The general treatment of massive quarks is documented in [3]. We here adapt this treatment to the case of massless kinematics, in the following way.

- Specify a mapping procedure that allows to translate an input parton state containing massive four-vectors for light-flavour quarks into an equivalent set of massless ones.
- Specify a reverse mapping that can be done at the end of the shower (or at an intermediate scale to change to a different number of massless flavours), to translate a set of massless partons into equivalent massive ones.
- Specify a procedure, consistent with the maps above, by which mass corrections can be applied in the context of the massless evolution.

A.11.1 Mapping from Massive to Massless Momenta

To map a set of partons containing massive four-vectors to equivalent massless ones, we map each 2-parton antenna in the input parton state to an equivalent massless one.

A.11.2 Mass Corrections

For a set of massless post-branching momenta, mass corrections are implemented in the following way, designed to fit with the mapping algorithm described above: first, identify the corresponding massive branching invariants by

$$s_{ij}^{\text{massless}} \rightarrow q_{ij}^2 = (2p_i \cdot p_j)^{\text{massive}} . \tag{118}$$

The equivalent massive phase-space boundaries can then be checked by requiring positivity of the Gram determinant [3]:

$$4\Delta_3 = q_{ij}^2 q_{jk}^2 q_{ik}^2 - q_{ij}^4 m_k^2 - q_{jk}^4 m_i^2 - q_{ik}^4 m_j^2 + 4m_i^2 m_j^2 m_k^2 \geq 0 , \tag{119}$$

with $m_{i,j,k}$ the would-be physical masses of the post-branching partons.

A further level of refinement can be obtained by modifying the singularity structure of the massless antenna functions to take universal eikonal mass corrections into account (exact for soft gluon emissions) [3]:

$$a_{\text{massive}}^{\text{eikonal}}(Q_i, g_j, \bar{Q}_k) = a_{\text{massless}} - \frac{2m_i^2}{q_{ij}^4} - \frac{2m_k^2}{q_{jk}^4}, \quad (120)$$

which can be applied as a multiplicative accept probability (with $P < 1$ since the corrections are negative) to all gluon-emission processes.

Finally, for branchings involving $g \rightarrow Q_j \bar{Q}_k$ splittings, we use the following multiplicative mass correction:

$$R_{Xg \rightarrow XQ\bar{Q}}(X_i, \bar{Q}_j, Q_k) = \frac{a_{\text{massless}} + \frac{m_Q^2}{m_{jk}^4}}{a_{\text{massless}}} \approx 1 + \frac{2m_Q^2}{m_{jk}^4} \frac{q_{IK}^4}{q_{ik}^4 + q_{ij}^4}. \quad (121)$$

Since the sign of the mass correction is positive here (opposite to the case for gluon emission above), a headroom factor slightly greater than unity may be required to accommodate the enhanced splitting probability within the trial splitting overestimates.

A.11.3 Mapping from Massless to Massive Momenta

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B Accept Probabilities

B.1 Helicity Selection

See [4].

B.2 Smooth-Ordering Factor: P_{imp}

Note: this section is largely adapted from the discussion in [5].

In smooth ordering, the only quantity which must still be strictly ordered are the antenna invariant masses, a condition which follows from the nested antenna phase spaces and momentum conservation. Within each individual antenna, and between competing ones, the measure of evolution time is still provided by the ordering variable, which we therefore typically refer to as the “evolution variable” in this context (rather than the “ordering variable”), in order to prevent confusion with the strong-ordering case. The evolution variable can in principle still be chosen to be any of the possibilities given above, though we shall typically use $2p_{\perp}$ for gluon emission and $m_{q\bar{q}}$ for gluon splitting.

In terms of an arbitrary evolution variable, Q , the smooth-ordering factor is [4]

$$P_{\text{imp}}(Q^2, \hat{Q}^2) = \frac{\hat{Q}^2}{\hat{Q}^2 + Q^2}, \quad (122)$$

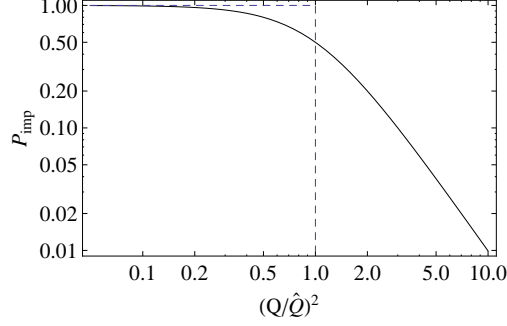


Figure 1: The smooth-ordering factor (*solid*) compared to a strong-ordering Θ function (*dashed*).

where Q is the evolution scale associated with the current branching, and \hat{Q} measures the scale of the parton configuration before branching. A comparison to the strong-ordering step function is given in fig. 1, on a log-log scale. Since this factor is bounded by $0 \leq P_{\text{imp}} \leq 1$, it can be applied as a simple accept/reject on each trial branching.

When switched on, smooth ordering is technically achieved as follows. After each accepted branching, the daughter antennae involved in that particular branching are allowed to restart their evolution from a scale nominally equivalent to their respective kinematic maximum. Trial branchings are then generated in the “unordered” part of phase space first, for those antennae only, while all other antennae in the event are “on hold”, waiting for the scale to drop back down to normal ordering before the global event evolution is continued. The P_{imp} factor is applied as an extra multiplicative modification to the accept probability for each trial branching, in both the ordered and unordered regions of phase space.

In the strongly-ordered region of phase-space, $Q \ll \hat{Q}$, we may rewrite the P_{imp} factor as

$$P_{\text{imp}} = \frac{1}{1 + \frac{Q^2}{\hat{Q}^2}} \stackrel{Q \ll \hat{Q}}{\approx} 1 - \frac{Q^2}{\hat{Q}^2} + \dots \quad (123)$$

Applying this to the $2 \rightarrow 3$ antenna function whose leading singularity is proportional to $1/Q^2$, we see that the overall correction arising from the Q^2/\hat{Q}^2 and higher terms is finite and of order $1/\hat{Q}^2$; a power correction. The LL singular behaviour is therefore not affected. However, at the multiple-emission level, the $1/\hat{Q}^2$ terms do modify the *subleading* logarithmic structure, starting from $\mathcal{O}(\alpha_s^2)$, as we shall return to below.

In the *unordered* region of phase-space, $Q > \hat{Q}$, we rewrite the P_{imp} factor as

$$P_{\text{imp}} = \frac{\hat{Q}^2}{Q^2} \frac{1}{1 + \frac{\hat{Q}^2}{Q^2}} \stackrel{Q > \hat{Q}}{\approx} \frac{\hat{Q}^2}{Q^2} \left(1 - \frac{\hat{Q}^2}{Q^2} + \dots \right), \quad (124)$$

which thus effectively modifies the leading singularity of the LL $2 \rightarrow 3$ function from $1/Q^2$ to $1/Q^4$, removing it from the LL counting. The only effective terms $\propto 1/Q^2$ arise from finite terms in the radiation functions, if any such are present, multiplied by the P_{imp} factor. Only a

matching to the full tree-level $2 \rightarrow 4$ functions would enable a precise control over these terms. Up to any given fixed order, this can effectively be achieved by matching to tree-level matrix elements. Matching beyond the fixed-order level is beyond the scope of the current treatment. We thus restrict ourselves to the observation that, at the LL level, smooth ordering is equivalent to strong ordering, with differences only appearing at the subleading level.

The effective $2 \rightarrow 4$ probability in the shower arises from a sum over two different $2 \rightarrow 3 \otimes 2 \rightarrow 3$ histories, each of which has the tree-level form

$$\hat{A} P_{\text{imp}} A \propto \frac{1}{\hat{Q}^2} \frac{\hat{Q}^2}{\hat{Q}^2 + Q^2} \frac{1}{Q^2} = \frac{1}{\hat{Q}^2 + Q^2} \frac{1}{Q^2}, \quad (125)$$

thus we may also perceive the combined effect of the modification as the addition of a mass term in the denominator of the propagator factor of the previous splitting. In the strongly ordered region, this correction is negligible, whereas in the unordered region, there is a large suppression from the necessity of the propagator in the previous topology having to be very off-shell, which is reflected by the P_{imp} factor. Using the expansion for the unordered region, eq. (124), we also see that the effective $2 \rightarrow 4$ radiation function, obtained from iterated $2 \rightarrow 3$ splittings, is modified as follows,

$$P_{2 \rightarrow 4} \propto \frac{1}{\hat{Q}^2} \frac{\hat{Q}^2}{Q^2} \frac{1}{Q^2} \rightarrow \frac{1}{Q^4} + \mathcal{O}(\dots), \quad (126)$$

in the unordered region. That is, the intermediate low scale \hat{Q} , is *removed* from the effective $2 \rightarrow 4$ function, by the application of the P_{imp} factor.

B.3 All-Orders P_{imp} Factor

The path through phase space taken by an unordered shower history is illustrated in fig. 2, from [5]. An antenna starts showering at a scale equal to its invariant mass, \sqrt{s} , and a first $2 \rightarrow 3$ branching occurs at the evolution scale \hat{Q} . This produces the overall Sudakov factor $\Delta_{2 \rightarrow 3}(\sqrt{s}, \hat{Q})$. A daughter antenna, produced by the branching, then starts showering at a scale equal to its own invariant mass, labeled $\sqrt{s_1}$. However, for all scales larger than \hat{Q} , the P_{imp} factor suppresses the evolution in this new dipole so that no leading logs are generated. To leading approximation, the effective Sudakov factor for the combined $2 \rightarrow 4$ splitting is therefore given by

$$\Delta_{2 \rightarrow 4}^{\text{eff}} \sim \Delta_{2 \rightarrow 3}(\sqrt{s}, \hat{Q}), \quad (127)$$

in the unordered region. Thus, we see that a dependence on the intermediate scale \hat{Q} still remains in the effective Sudakov factor generated by the smooth-ordering procedure. Since $\hat{Q} < Q$ in the unordered region, the effective Sudakov suppression of these points might be “too strong”. The smooth ordering therefore allows for phase space occupation in regions corresponding to dead zones in a strongly ordered shower, but it does suggest that a correction to the Sudakov factor may be desirable, in the unordered region.

A study of $Z \rightarrow 4$ jets at one loop would be required to shed further light on this question. In the meantime, for all unordered branchings that follow upon a gluon emission, we allow to

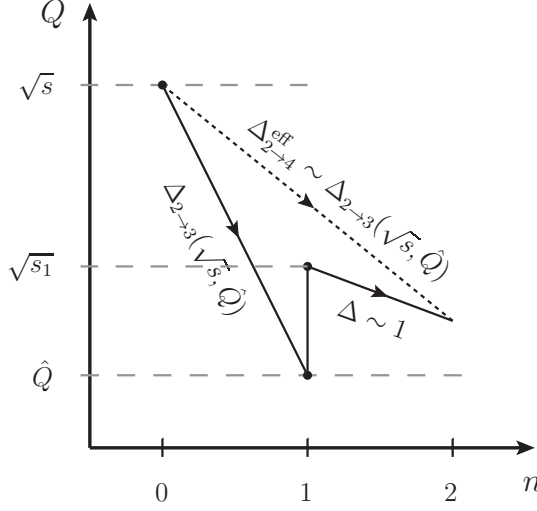


Figure 2: Illustration of scales and Sudakov factors involved in an unordered sequence of two $2 \rightarrow 3$ branchings, representing the smoothly ordered shower's approximation to a hard $2 \rightarrow 4$ process.

include a correction to the P_{imp} factor that removes the leading (eikonal) part of the “extra” Sudakov suppression. We define an all-orders corrected P_{imp} factor as follows:

$$P_{\text{imp}}^{\text{emit}}(Q^2, \hat{Q}^2) \rightarrow \frac{\alpha_s(Q^2)}{\alpha_s(\hat{Q}^2)} \frac{P_{\text{imp}}(Q^2, \hat{Q}^2)}{\Delta_{2 \rightarrow 3}^{\text{eik}}(Q^2, \hat{Q}^2)}, \quad (128)$$

with the Eikonal terms of the Sudakov integral given by [5]:

$$\frac{1}{\Delta_{2 \rightarrow 3}^{\text{eik}}(Q^2, \hat{Q}^2)} = \exp \left(\frac{\alpha_s(Q^2)}{2\pi} \mathcal{C} [I_1(\hat{y}) - 2I_2(\hat{y}) - I_1(y) + 2I_2(y)] \right),$$

where \mathcal{C} is the colour factor of the first $2 \rightarrow 3$ branching (the one that produced the intermediate scale \hat{Q}) inside which the unordered $2 \rightarrow 4$ branching is occurring, and $y = Q^2/m_2^2$ ($\hat{y} = \hat{Q}^2/m_2^2$) is the branching scale normalized to the invariant mass squared of that antenna. For evolution in p_\perp (the default for gluon emissions), the I_1 and I_2 integrals are [5]:

$$I_1(y) = \frac{\pi^2}{6} + \frac{1}{2} \ln^2 \left[\frac{y^2}{2(1 + \sqrt{1 - y^2}) - y^2} \right] - \ln^2 \left[\frac{1}{2} (1 + \sqrt{1 - y^2}) \right] - 2 \text{Li}_2 \left[\frac{1}{2} (1 + \sqrt{1 - y^2}) \right] \quad (129)$$

$$I_2(y) = - \left(\ln \left[\frac{y^2}{2(1 + \sqrt{1 - y^2}) - y^2} \right] + 2\sqrt{1 - y^2} \right), \quad (130)$$

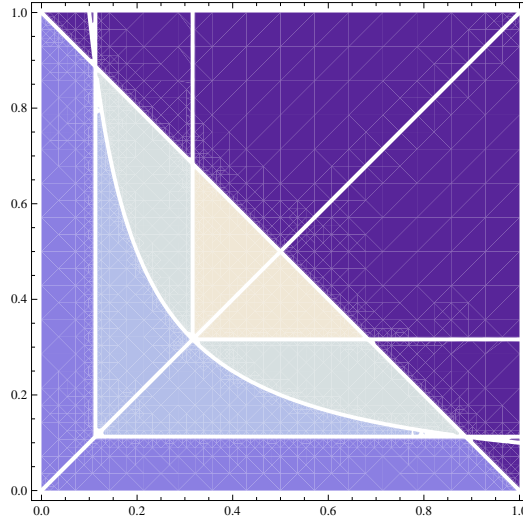


Figure 3: Illustration of the intersection/nesting of p_T and m_D contours.

with expressions for other ordering types available in [5]. We note that this factor neglects (positive) collinear-singular terms and (positive) corrections from the running of α_s between Q and \hat{Q} , hence we expect that even this correction factor still only represents a partial compensation, at a level equivalent to removing spurious terms of total order $\alpha_s^3 \ln^3(\hat{Q}^2/Q^2)$ and $\alpha_s^3 \ln^2(\hat{Q}^2/Q^2)$. We also note that a similar factor could be applied

B.4 Gluon Splitting: The Ariadne Factor

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B.5 Matrix-Element Corrections: Leading Colour

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B.6 Matrix-Element Corrections: Full Colour

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B.7 Matrix-Element Corrections: Different Interfering Borns

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C Cutoff Boundaries

C.1 Fixed Transverse Momentum

- Consider the region defined by $y_{ij}y_{jk} \geq y_{\perp}$. For illustration, a value of $y_{\perp} = 0.1$ was used for the contour shown with light green shading in fig. 3.
- The (larger) invariant-mass region that completely encloses the y_{\perp} one is defined by $y_D = \min(y_{ij}, y_{jk}) \geq \frac{1}{2}(1 - \sqrt{1 - 4y_{\perp}})$. This is shown with light blue shading in fig. 3.
- The (smaller) invariant-mass region that is completely enclosed by the y_{\perp} one is defined by $y_D = \min(y_{ij}, y_{jk}) \geq \sqrt{y_{\perp}}$. This is shown with light yellow shading in fig. 3.

To translate this to evolution variables, with arbitrary normalization factors, use $y_{\perp} = Q_{\perp}^2/s_{IK}/N_{\perp}$ and $m_D^2/s_{IK}/N_D$.

C.2 Fixed Dipole Mass

- Consider the region defined by $\min(y_{ij}y_{jk}) \geq y_D$, with y_D some fixed value.
- The (larger) transverse-momentum region that completely encloses the y_D one is defined by $y_{\perp} = y_{ij}y_{jk} \geq y_D^2$. This relationship is illustrated by the light-green and light-yellow shaded regions in fig. 3.
- The (smaller) transverse-momentum region that is completely enclosed by the y_D one is defined by $y_{\perp} = y_{ij}y_{jk} \geq \frac{1}{4}(1 - (1 - 2y_D)^2)$. This relationship is illustrated by the light-green and light-blue shaded regions in fig. 3.

To translate this to evolution variables, with arbitrary normalization factors, use $y_{\perp} = Q_{\perp}^2/s_{IK}/N_{\perp}$ and $m_D^2/s_{IK}/N_D$.

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